Quiz 9 Solutions

(1) *Find the length of the curve $y = \ln(\sec(x))$ for $0 \le x \le \pi/4$.*

Since $dy/\,dx = \sec(x)\tan(x)/\sec(x) = \tan(x)$, we have

$$
\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1 + \tan^2(x)}\ dx \\
&= \int_0^{\pi/4} \sec x\ dx \\
&= \ln|\sec x + \tan x|\big]_0^{\pi/4} \\
&= \ln(\sqrt{2} + 1).
\end{aligned}
$$

(2) *Evaluate the following integral or show that it is divergent.*

$$
\int_0^4 \frac{\ln x}{\sqrt{x}}\ dx.
$$

First, integrating by parts gives that

$$
\int \frac{\ln x}{\sqrt{x}}\ dx = 2\sqrt{x}\ln(x) - 4\sqrt{x}.
$$

Since $\lim_{x\to 0} \frac{\ln x}{\sqrt{x}} = \infty$, the definite integral is improper at $x = 0$. So

$$
\begin{aligned}
\int_0^4 \frac{\ln x}{\sqrt{x}}\ dx &= \lim_{a\to 0^+} \int_a^4 \frac{\ln x}{\sqrt{x}}\ dx \\
&= \lim_{a\to 0^+} 4\ln 4 - 8 - 2\sqrt{a}\ln a - 4\sqrt{a} \\
&= 4\ln 4 - 8 - 2 \lim_{a\to 0^+} \sqrt{a}\ln a.
\end{aligned}
$$

This last limit is a $(-\infty)\cdot 0$ indefinite form, so we use L'Hopital's rule:

$$
\begin{aligned}
\lim_{a\to 0^+} \sqrt{a}\ln a &= \lim_{a\to 0^+} \frac{\ln a}{a^{-1/2}} \\
&= \lim_{a\to 0^+} \frac{1/a}{-\frac{1}{2}a^{-3/2}} \\
&= \lim_{a\to 0^+} -2\sqrt{a} = 0.
\end{aligned}
$$

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So we conclude that
$$\int_0^4 \frac{\ln x}{\sqrt{x}} \, dx = 4\ln 4 - 8.$$

(3) *Find*
$$\int \frac{1}{\sqrt{x + x^{3/2}}} \, dx.$$

Using the substitution $u = 1 + \sqrt{x}$ (and $2\,du = 1/\sqrt{x}\,dx$), we get
$$
\begin{aligned}
\int \frac{dx}{\sqrt{x + x^{3/2}}} &= \int \frac{dx}{\sqrt{x}\sqrt{1 + \sqrt{x}}} \\
&= 2\int \frac{du}{\sqrt{u}} \\
&= 4\sqrt{u} \\
&= 4\sqrt{1 + \sqrt{x}} + C.
\end{aligned}
$$