Math 54, Spring 2009, Sections 109 and 112
Midterm 2 Review

This sheet mentions a lot of the major ideas from Chapters 4, 5 and 6. It is inevitably inexhaustive, but hopefully it can help you notice some areas where you might need to review some more.

$\mathbb{R}^n$ vs. Vector spaces

- In $\mathbb{R}^n$, we had defined operators of scalar multiplication and addition. Inspired by this, we defined a **vector space** to be any set of objects that have addition and scalar multiplication operations that behave like those in $\mathbb{R}^n$, with the full list of axioms given on p.217.
- We can then define the concepts of **subspaces**, **spanning** and **linear independence** the same way we did for $\mathbb{R}^n$. Note: if H is a subspace of V, then H is again a vector space, with the same operations as V.
- Just like with vector spaces, a **basis** is a linearly independent spanning set. However, not all vector spaces have finite bases. A vector space with a finite basis is called **finite-dimensional**. All bases for a given finite-dimensional vector space have the same number of elements.
- Any linearly independent set in a vector space can be expanded to a basis by adding more elements. Any spannig set can be contracted to a basis by removing redundent elements. To do this, order your spanning set, and keep removing vectors that can be written as linear combinations of the ones before.
- Informally speaking, any (finite-dimensional) vector space with dimension n looks and feels like $\mathbb{R}^n$. What is the formal version of this statement? If $\mathcal{B}$ is a basis for V, then the coordinate map $[\cdot]_\mathcal{B}$ is an invertible linear transformation (**isomorphism**) between V and $\mathbb{R}^n$. This isomorphism can be used to prove that V shares many of the same properties as $\mathbb{R}^n$ (p.250-251).
- If $\mathcal{B}$ and $\mathcal{C}$ are different bases for V, we may be interested in the relationship between $[\vec{x}]_\mathcal{B}$ and $[\vec{x}]_\mathcal{C}$. For any pair of bases, there is a unique, invertible matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{x}]_\mathcal{B} = [\vec{x}]_\mathcal{C}$ for all $\vec{x} \in V$ (p.273).

- More generally, if $T : V \to W$ is a linear transformation, $\mathcal{B}$ is a basis for V, and $\mathcal{C}$ is a basis for W, then there is a unique matrix M such that $M[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{C}}$ (p. 329). If $V = W$, then $M = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$. Since V and W are just $\mathbb{R}^n$ and $\mathbb{R}^m$ in disguise, you can think of M as doing the same thing as T, just on the undisguised versions of V and W.

- If $V = W$ and $\mathcal{B} = \mathcal{C}$ in the previous bullet point, then the matrix M is called $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix of T. Note: this is the same notation as coordinates, but this is different; this does not mean we are taking the coordinates of a matrix. However, we do have $[T(\vec{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$ if $T : V \to V$ is a linear transformation.

Eigenvectors and eigenvalues

- An **eigenvector/eigenvalue** pair for a matrix A is a non-zero vector x and a scalar λ such that $A\vec{x} = \lambda\vec{x}$. The **eigenspace** of a matrix A with respect to the eigenvalue λ is the set of all eigenvectors of A with eigenvalue λ, along with the zero vector. Alternatively, it is the subspace $\mathrm{Nul}(A - \lambda I)$.

- If a matrix is **triangular** (or diagonal), the eigenvalues are the entries on the diagonal. If not, you can find the eigenvalues by finding the roots of the **characterisic polynomial** $\det(A - \lambda I)$.

- **Similar** matrices have the same eigenvalues. A and A^t have the same eigenvalues. (Can you prove these things?)

- If A is $n \times n$, and the dimensions of the eigenspaces of A add up to n, then A is **diagonalizable** (Theorem 7, p.324). That is, there is an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

- If an $n \times n$ matrix has n different eigenvalues, then it is diagonalizable (since every eigenspace has dimension at least 1). However, the converse is not true. The matrix $2I$ has only one eigenvalue, 2, but it is diagonal(izable).

Orthogonality and related ideas

- The existence of an inner product (the dot product) on $\mathbb{R}^n$ lets us define the notions of **orthogonal vectors** (where $\vec{x} \cdot \cdot \cdot \vec{y} = 0$) and **norm** of vectors $\|x\| = \sqrt{\vec{x} \cdot \vec{x}}$. If $\vec{x} \cdot \vec{y} = 0$, then $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ (the Pythagorean Theorem in n-dimensions).

- A set S is called **orthogonal** if $\vec{x}$ and $\vec{y}$ are orthogonal for every pair of distinct vectors $\vec{x}, \vec{y} \in S$. Every orthogonal set is linearly independent.

- We're particularly interested in **orthogonal bases** and **orthonormal bases** (where $\|\vec{x}\| = 1$ for every basis vector. Note: you can turn an orthogonal basis into an orthonormal basis by dividing every basis vector by its length). To turn an ordinary basis into an orthogonal basis, use **Gram-Schmidt** (p.402-).

- If W is a subspace of $\mathbb{R}^n$, then we define the **orthogonal complement** $W^\perp$ to be everything in $\mathbb{R}^n$ that is orthogonal to everything in W. Given any vector $\vec{y} \in \mathbb{R}^n$, it can be written uniquely in the form $\vec{y} = \hat{y} + \vec{z}$, where $\operatorname{Proj}_W \vec{y} = \hat{y} \in W$ and $\vec{z} \in W^\perp$. This can be calculated via Theorem 8 (p. 395) if you have an orthogonal basis for W.

- The vector $\hat{y}$ from the previous bullet is the **closest point** in W to $\vec{y}$ (Theorem 9, p.398).

- One use of the previous fact is that it allows us to find $\vec{x}$ that makes $\left\|A\vec{x} - \vec{b}\right\|$ as small as possible for a given matrix A and $\vec{b}$. If $A\vec{x} = \vec{b}$ is consistent, then we just want to solve $A\vec{x} = \vec{b}$. If not, calculate $\operatorname{Proj}_{\operatorname{Col} A} \vec{b}$, and solve $A\vec{x} = \operatorname{Proj}_{\operatorname{Col} A} \vec{b}$ instead (p.414). Alternatively, one can solve the **normal equations** $A^T A\vec{x} = A^T\vec{b}$ (p.411).

- Just as we have generalized many notions from $\mathbb{R}^n$ to vector spaces in general, we define the notion of **inner product** on a vector space to be anything that has some of the same properties as the dot product (p.428 for the list of axioms).

- This allows us to define length and orthogonality in a vector space. However, these concepts depend on the particular inner product chosen. In general, there are infinitely different inner products that can be defined on a single vector space, so there is no "correct" notion of length or orthogonality on a given vector space unless there is a "correct" or "standard" inner product for that vector space (like the dot product for $\mathbb{R}^n$, which gives us the expected notions of length and orthogonality based on our intuition regarding the world around us).

- Given an inner product $\langle \cdot, \cdot \rangle$ on a vector space V, and $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$, we have the following two inequalities (p432-433):

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|x\| \, \|y\| \,, \qquad \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \,.$$

Some things you should know how to do

- Determine if a set of vectors in a vector space V is linearly independent (writing vectors as a linear combinations of the ones before, or using coordinates).
- Given two bases for a vector space, find the corresponding change of basis matrix (p.273 onward).
- Find the eigenvalues of a matrix (p.313)
- Find (orthogonal) bases for the eigenspaces of a matrix
- Determine if a matrix is diagonalizable and if possible, diagonalize it (via the last two steps).
- Gram-Schmidt
- Find least-squares solutions to systems of linear equations
- Compute the projections of vectors onto subspaces.