Name: ______________________________

Math 54, Spring 2009, Section 112
Quiz 3 Solutions

[**1** - (3 pts)] Let V be the vector space $\mathbb{P}_2$ of polynomials of degree at most 2, with bases $\mathcal{B} = \{x^2 - 2x + 2, x, x^2 + 2x - 1\}$ and $\mathcal{C} = \{x^2, x^2 + x, x - 1\}$. Find the change-of-basis matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$. (Hint: try to write elements of $\mathcal{B}$ in terms of the elements of $\mathcal{C}$).

By inspection, $\vec{b}_1 = \vec{c}_1 - 2\vec{c}_3$, $\vec{b}_2 = -\vec{c}_1 + \vec{c}_2$, and $\vec{b}_3 = \vec{c}_2 + \vec{c}_3$. Thus $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$.

We can find $\underset{B\leftarrow C}{P}$ using $\underset{B\leftarrow C}{P} = (\underset{C\leftarrow B}{P})^{-1}$, so row reduce

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{array}\right].$$

Thus $\underset{B\leftarrow C}{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

[**2** - (3 pts)] Suppose you solve a non-homogenous system of 8 linear equations in 10 unknowns, and find that in your solution you have exactly 2 free variables. Is it possible to change the right-hand side of the system of equations (i.e. "change $\vec{b}$") and have the resulting system be inconsistent? (No points for correct answer with incorrect reasoning.)

This situation corresponds to a matrix equation $A\vec{x} = \vec{b}$, where A is 8×10 and has 2 non-pivot columns (i.e. 2 free variables). Thus A has 8 pivots, and $\operatorname{Rank} A = \dim \operatorname{Col} A = 8$. Since $\operatorname{Col} A \subseteq \mathbb{R}^8$, this means that $\operatorname{Col} A = \mathbb{R}^8$. Thus $A\vec{x} = \vec{c}$ is consistent for evey $\vec{c} \in \mathbb{R}^8$.

[**3** - (3 pts)] Let V be a vector space with basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$. Prove **exactly one** of the following two things (if you do both, I'll just grade the first):

(a) If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent, then so is $\{[\vec{v}_1]_{\mathcal{B}}, \ldots, [\vec{v}_k]_{\mathcal{B}}\}$.

(b) If $\vec{v} \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$, then $[\vec{v}]_{\mathcal{B}} \in \text{Span}\{[\vec{v}_1]_{\mathcal{B}}, \ldots, [\vec{v}_k]_{\mathcal{B}}\}$.

(a) Assume that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent, and that $x_1[\vec{v}_1]_{\mathcal{B}} + \cdots + x_k[\vec{v}_k]_{\mathcal{B}} = \vec{0}$. We need to show that $x_1 = \cdots = x_k = 0$. Using the linearity of the coordinate transformation, we have $[x_1\vec{v}_1 + \cdots + x_k\vec{v}_k]_{\mathcal{B}} = \vec{0}$. Here we could use the fact that $[\cdot]_{\mathcal{B}}$ is one-to-one, but instead we compute directly from the definition of coordinates that

$$x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = 0\vec{b}_1 + \cdots + 0\vec{b}_n = \vec{0}.$$

But our original set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent, so we must have $x_1 = \cdots = x_k = 0$, which was to be shown.

(b) If $\vec{v} \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$, then there are weights $x_1, \ldots, x_k$ such that $\vec{v} = x_1\vec{v}_1 + \cdots + x_k\vec{v}_k$. Taking coordinates of both sides, and using the fact that the coordinate transformation is linear, we get

$$\begin{aligned} [\vec{v}]_{\mathcal{B}} &= [x_1\vec{v}_1 + \cdots + x_k\vec{v}_k]_{\mathcal{B}} \\ &= x_1[\vec{v}_1]_{\mathcal{B}} + \cdots + x_k[\vec{v}_k]_{\mathcal{B}}. \end{aligned}$$

Thus $[\vec{v}]_{\mathcal{B}} \in \text{Span}\{[\vec{v}_1]_{\mathcal{B}}, \ldots, [\vec{v}_k]_{\mathcal{B}}\}$.