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Math 54, Spring 2009, Section 112
Quiz 5 Solutions

[**1** - (5 pts)] a) Find the general solution of the differential equation $y'' - y' - 2y = t^2 - e^{-t}$.
b) Find the solution of the ODE from part (a) with initial values $y(0) = \frac{1}{4}$ and $y'(0) = -\frac{5}{12}$.

(a) The auxiliary equation is $0 = r^2 - r - 2 = (r-2)(r+1)$, so the general solution to the homogenous equation $y'' - y' - 2y = 0$ is $y_h = c_1 e^{2t} + c_2 e^{-t}$. We now try to solve $y'' - y' - 2y = t^2$ by guessing $y_{p_1} = At^2 + Bt + C$ (no extra factor of t, because 0 is not a root of the auxiliary equation). Plugging it in, we get

$$t^2 = y_{p_1}'' - y_{p_1}' - 2y_{p_1} = -2A - 2At - B - 2At^2 - 2Bt - 2C = -2At^2 - 2(A+B)t + 2A - B - 2C.$$

Solving gives $A = -\frac{1}{2}$, $B = \frac{1}{2}$, and $C = -\frac{3}{4}$. Next we guess a solution y_{p_2} to $y'' - y' - 2y = -e^{-t}$. Since -1 is a (single) root of the auxiliary equation, we multiply our normal guess by t to get $y_{p_2} = Dte^{-t}$. Plugging it in,

$$-e^{-t} = y_{p_2}'' - y_{p_2}' - 2y_{p_2} = D(t-2)e^{-t} + D(t-1)e^{-t} - 2Dte^{-t} = -3De^{-t},$$

so $D = \frac{1}{3}$. Thus the general solution to the ODE is

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{3}{4} + \frac{1}{3}te^{-t}.$$

(b) Plugging in the initial values gives $\frac{1}{4} = y(0) = c_1 + c_2 - \frac{3}{4}$, and $-\frac{5}{12} = y'(0) = -c_1 + 2c_2 + \frac{1}{2} + \frac{1}{3}$, which gives the system of equations $c_1 + c_2 = 1$ and $2c_1 - c_2 = -\frac{15}{12}$. Solving gives $c_1 = -\frac{1}{12}$ and $c_2 = \frac{13}{12}$, so the solution to the initial value problem is

$$y(t) = -\frac{1}{12}e^{2t} + \frac{13}{12}e^{-t} - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{3}{4} + \frac{1}{3}te^{-t}.$$

[**2** -(4 pts)] Let $L[y]$ be a linear differential operator, and consider the ODE $L[y] = g(x)$. Suppose that y_1 and y_2 are solutions of this ODE on the whole line $(-\infty, \infty)$. Show that if $y_1 + y_2$ is also a solution on the whole line, then the equation is homogenous (i.e. $g(x) = 0$ for all x).

If y_1, y_2 and y_1+y_2 are all solutions to the *linear* ODE $L[y] = g(x)$, we have $g(x) = L[y_1+y_2] = L[y_1] + L[y_2] = g(x) + g(x)$. Subtracting a $g(x)$ from both sides gives $g(x) = 0$ for all x.