Math 54, Spring 2009, Sections 109 and 112
Worksheet 4 (Lay 4.1-4.3)
Solutions

(1) Let V be the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Is the set $\{\sin x, \cos x, e^x\}$ linearly independent? Find a basis for $\text{Span}\{\sin x, \cos x, e^x\}$.

We need to check if any of the elements is a linear combination of the ones before it. If $\cos x = c_1 \sin x$, then by plugging in $x = 0$ we would have $0 = 1$, a contradiction. So $\cos x$ is not a multiple of $\sin x$. Now suppose $e^x = c_1 \cos x + c_2 \sin x$. Then plugging in 0 and 2π we get $1 = c_1$ and $e^{2\pi} = c_1$. These can't both be possible, so e^x is not a linear combination of cos and sin. Thus the set is linearly independent, and $\{\sin x, \cos x, e^x\}$ is a basis for $\text{Span}\{\sin x, \cos x, e^x\}$.

(2) True or False? If true, justify. If false, give a counterexample. In these statements, V is a vector space, and H is a subspace of V.

(a) If $\vec{u} \in H$ and $\vec{v} \in H$, then $\text{Span}\{\vec{u}, \vec{v}\} \subseteq H$.

(b) Some basis for $\mathbb{P}_n$ (polynomials of degree at most n) has n elements.

(c) If a finite set S of non-zero vectors spans V, then some subset of S is a basis for V.

(d) A linear transformation is one-to-one if and only if $\text{Kernel}(T) = \{0\}$.

(a) True. If $\vec{u} \in H$ and $\vec{v} \in H$, then $c_1\vec{u} + c_2\vec{v} \in H$ for any scalars c_1, c_2 (by the definition of subspace). So H contains every linear combination of $\vec{u}$ and $\vec{v}$, so $\text{Span}\{\vec{u}, \vec{v}\} \subseteq H$.

(b) False. One basis for $\mathbb{P}_n$ is $\{1, t, t^2, \ldots, t^n\}$, which has $n + 1$ elements. All bases for a given space have the same number of elements, so no basis for $\mathbb{P}_n$ has n elements.

(c) True. See the Spanning Set Theorem, p.239.

(d) True. We've seen the analagous statement for matrices, that $A\vec{x} = \vec{b}$ has at most one solution for each $\vec{b}$ if and only if $\operatorname{Nul} A = \{\vec{0}\}$. To prove the statement, we need to prove both directions. First assume that T is one-to-one. That is, assume that if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$ (so that no two different inputs can be sent to the same output). Now suppose that $x \in \operatorname{Ker} T$. Then $T(\vec{x}) = \vec{0} = T(\vec{0})$. Since T is one-to-one, this means that $\vec{x} = \vec{0}$. This means that any arbitrary element of $\operatorname{Ker} T$ must be the zero vector, so $\operatorname{Ker} T = \{\vec{0}\}$.

Conversely, suppose that $\operatorname{Ker} T = \{\vec{0}\}$, and that $T(\vec{x}) = T(\vec{y})$. We would like to show that $\vec{x} = \vec{y}$ (so that T would be one-to-one). Subtracting $T(\vec{y})$ from both sides, and using the linearity of T, we get $T(\vec{x} - \vec{y}) = \vec{0}$. So $\vec{x} - \vec{y} \in \operatorname{Ker} T$ by the definition of $\operatorname{Ker} T$. But $\operatorname{Ker} T$ contains only the zero vector, so $\vec{x} - \vec{y} = \vec{0}$. Thus $\vec{x} = \vec{y}$, which completes the proof that T is one-to-one.

(3) Let $M_{n\times m}(\mathbb{R})$ be the vector space of $n \times m$ matrices. Define $T : M_{2\times 3}(\mathbb{R}) \to M_{2\times 3}(\mathbb{R})$ by $T(A) = AB$, where $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 4 & 5 \end{bmatrix}$ is fixed. Show that T is one-to-one and onto (i.e. find Range(T) and Kernel(T)).

Note that T is not a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$, so we cannot find its standard matrix. To show that T is one-to-one and onto, we must show that $\operatorname{Ker} T = \{0\}$ and that $\text{Range}(T) = M_{2\times 3}(\mathbb{R})$. If $T(A) = 0$, then $AB = 0$. Since $\det(B) = -20$, B is invertible. Multiple both sides of the previous equality on the right by B^{-1} to get $A = 0$. Thus if $T(A) = 0$, then we must have $A = 0$. So $\operatorname{Ker} T = \{0\}$ and T is one-to-one.

Now we want to show that given any $C \in M_{2\times 3}(\mathbb{R})$, there is some input that will have C as an output (i.e. that T is onto). We'd like $T(A) = C$, or in other words $AB = C$. For that to happen, we'd need $A = CB^{-1}$. Let's try it: $T(CB^{-1}) = CBB^{-1} = C$. So $C \in \text{Ran}(T)$. Since C was arbitrary, $\text{Ran}(T) = M_{2\times 3}(\mathbb{R})$ and T is onto.

(4) Let V be the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ that also have a continuous derivative, and let W be the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define $T : V \to W$ by $T(f) = f'$. Justify why V and W are vector spaces, and why T is a linear transformation. What is $\operatorname{Ker} T$? Bonus: use calculus to show that T is onto.

Both V and W are subsets of the vector space of all functions from $\mathbb{R} \to \mathbb{R}$, so we just need to explain why they are subspaces. The function $f(x) = 0$ is continuous and differentiable, so both V and W have the zero vector. The sum of continuous functions is continuous and any scalar multiple of a continuous functions is continuous, so W is a subspace. Also, if f and g are differentiable, so is $f + g$, with $(f + g)' = f' + g'$. Also, so is cf, with $(cf)' = cf'$. The last two statements justify why V is a vector space, and why T is linear.

To find $\operatorname{Ker} T$, suppose that $T(f) = \vec{0}$. That is, $f' = 0$. If the deriviative of a function is the constant zero function, then f must be constant. So $\operatorname{Ker} T$ is the set of all constant functions.

To show that T is onto, fix a continuous function $f \in W$. We need to show that there is some input that yields f. That is, we need some $F \in V$ such that $F' = f$. Let $F(x) = \int_0^x f(t)dt$. By the Fundamental Theorem of Calculus, $F \in V$ (i.e. F is differentiable) and $F' = f$. So $T(F) = f$, and since f was arbitrary, T is onto.