Math 54, Spring 2009, Sections 109 and 112
Worksheet 5 (Lay 4.4-4.7)
Solutions

(1) (p.276, #6) Let $\mathcal{D} = \{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ and $\mathcal{F} = \{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ be bases for a vector space V, and suppose $\vec{f}_1 = 2\vec{d}_1 - \vec{d}_2 + \vec{d}_3$, $\vec{f}_2 = 3\vec{d}_2 + \vec{d}_3$, and $\vec{f}_3 = -3\vec{d}_1 + 2\vec{d}_3$. Find the change-of-coordinate matrix from $\mathcal{F}$ to $\mathcal{D}$. Find $[\vec{x}]_{\mathcal{D}}$ for $\vec{x} = \vec{f}_1 - 2\vec{f}_2 + 2\vec{f}_3$.

By inspection, $[\vec{f}_1]_{\mathcal{D}} = (2, -1, 1)$, $[\vec{f}_2]_{\mathcal{D}} = (0, 3, 1)$ and $[\vec{f}_3]_{\mathcal{D}} = (-3, 0, 2)$. So by Theorem 15 (p.273), we have

$$\underset{\mathcal{D}\leftarrow\mathcal{F}}{P} = \begin{bmatrix} [\vec{f}_1]_{\mathcal{D}} & [\vec{f}_2]_{\mathcal{D}} & [\vec{f}_3]_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

For $\vec{x} = \vec{f}_1 - 2\vec{f}_2 + 2\vec{f}_3$, we have $[\vec{x}]_{\mathcal{F}} = (1, -2, 2)$, so

$$[\vec{x}]_{\mathcal{D}} = \underset{\mathcal{D}\leftarrow\mathcal{F}}{P}[\vec{x}]_{\mathcal{F}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 3 \end{bmatrix}.$$

(2) True or False? Justify your answer.

(a) If $\mathcal{B}$ and $\mathcal{C}$ are different finite bases for V, then $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ can be singular (recall that singular means "not invertible").

(b) Let H be a subspace of a finite-dimensional vectors space V, and let $\mathcal{B} = \{b_1, \dots, b_r\}$ be a basis for V. Then $H = V$ if and only if $\mathcal{B} \subset H$.

(c) If P is an invertible $n \times n$ matrix, then there are bases $\mathcal{B}$ and $\mathcal{C}$ for $\mathbb{R}^n$ such that $P = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

(a) False, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is always invertible. Recall that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_m]_{\mathcal{C}} \end{bmatrix}$. Because $\mathcal{B}$ is a basis, its elements are linearly independent and span V. But then $\{[\vec{b}_1]_{\mathcal{C}}, \dots [\vec{b}_m]_{\mathcal{C}}\}$ is a basis for $\mathbb{R}^m$ (because the coordinate map is an isomorphism). By the Invertible Matrix Theorem, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is invertible.

(b) True. If $H = V$, then we must have $\mathcal{B} \subset H$ (as the elements of a basis for V must be in V). On the other hand, if $\mathcal{B} \subset H$, then $\operatorname{Span} B \subseteq H$ because H is a subspace. But $\operatorname{Span} B = V$, so $V \subseteq H$. By the definition of subspace, $H \subseteq V$. The only way this is possible is if $V = H$. (This can also be done by showing that $\dim V = \dim H$.)

(c) True. Let $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be the columns of P (which is a basis for $\mathbb{R}^n$ by the Invertible Matrix Theorem), and let $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then $P = P_{\mathcal{B}}$ by the definition of $P_{\mathcal{B}}$. As on p.274, we have $P_{\mathcal{B}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$.

(3) Let A be an $n \times n$ matrix, and let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. Find a formula for the matrix C such that $C[\vec{x}]_{\mathcal{B}} = [A\vec{x}]_{\mathcal{B}}$.

Recall that $P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x}$. Thus the condition $C[\vec{x}]_{\mathcal{B}} = [A\vec{x}]_{\mathcal{B}}$ for all $\vec{x}$ is equivalent to $CP_{\mathcal{B}}^{-1}\vec{x} = P_{\mathcal{B}}^{-1}A\vec{x}$ for all $\vec{x}$, which in turn is equivalent to $CP_{\mathcal{B}}^{-1} = P_{\mathcal{B}}^{-1}A$. Solving for C gives $C = P_{\mathcal{B}}^{-1}AP_{\mathcal{B}}$.

Note: the condition $C = P^{-1}AP$ says that A and C are *similar* matrices, an idea which we will explore more in Chapter 5. It means that C and A behave in very much the same way, but they are acting with respect to different coordinate systems.

(4) (p. 299, # 9) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. What are the dimensions of the range and kernel of T if T is one-to-one? What about if T is onto?

If T is one-to-one, that means that $\operatorname{Kernel} T = \{0\}$ so $\dim \operatorname{Kernel} T = 0$. If A is the standard matrix of T, this means that $\dim \operatorname{Nul} A = 0$. By the Rank Theorem (p.265), this means that $\operatorname{Rank} A = \dim \operatorname{Col} A = n$ (note: A is $m \times n$, so n is the number of columns). But $\operatorname{Col} A = \operatorname{Range} T$, so $\dim \operatorname{Range} T = n$.

On the other hand, if T is onto then $\operatorname{Rank} A = m$, so $\dim \operatorname{Kernel} T = \dim \operatorname{Nul} A = n - m$.