Name: ______________________________

Math 54, Summer 2009, Lecture 4
Quiz 3 Solution

(1) (a, *3 pts*) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with the property that

$$T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}4\\-6\end{bmatrix}, \qquad T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}2\\2\end{bmatrix}.$$

Find the standard matrix of T.

(b, *2 pts*) Is T one-to-one? Is T onto?

(a) To find the standard matrix of T, we need to know $T(\vec{e}_1)$ and $T(\vec{e}_2)$. Solving vector equations, we find that

$$\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}.$$

Hence the first column of the standard matrix of T is

$$\begin{aligned} T(\begin{bmatrix}1\\0\end{bmatrix}) &= T(\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}) \\ &= \frac{1}{2}T(\begin{bmatrix}1\\1\end{bmatrix}) + \frac{1}{2}T(\begin{bmatrix}1\\-1\end{bmatrix}) \\ &= \frac{1}{2}\begin{bmatrix}4\\-6\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\2\end{bmatrix} \\ &= \begin{bmatrix}3\\-2\end{bmatrix}. \end{aligned}$$

Similarly, one can solve a vector equation to find that

$$\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}.$$

Using a similar method to before, we can use this to calculate the second column of the standard matrix of T,

$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{2}T(\begin{bmatrix}1\\1\end{bmatrix}) - \frac{1}{2}T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}1\\-4\end{bmatrix}.$$

Thus the standard matrix of T is $\begin{bmatrix}3 & 1\\-2 & -4\end{bmatrix}$.

(b) Row reducing the standard matrix of T, we find that it has a pivot in every row and every column:

$$\begin{bmatrix} 3 & 1 \\ -2 & -4 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ -6 & -12 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & -10 \end{bmatrix}.$$

Thus by one of our theorems, T is both one-to-one and onto.

(2) (a, *2 pts*) Suppose that $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^p \to \mathbb{R}^q$ are linear transformations. Assuming that $S \circ T$ is defined, and that both S and T are one-to-one, what can you say about the equality/relative size of n, m, p, and q?

(b, 2 pts) Show that $S \circ T$ is also one-to-one (Hint: a linear transformation L is one-to-one if and only if $L(\vec{x}) = \vec{0}$ has only the trivial solution).

(a) For $(S \circ T)(\vec{x}) = S(T(\vec{x}))$ to be defined, the codomain (output) of T must match the domain (input) of S, so we must have $n = q$.

Since S is one-to-one, its standard matrix must have a pivot in every column. Since this matrix is $m \times n$, it has m rows and n columns. Having a pivot in every column is impossible unless there are at least as many rows as columns, so $m \geq n$. Similarly, because T is one-to-one we must have $q \geq p$. So the full summary is $m \geq n = q \geq p$.

(b) Approach 1 (following the hint): We must show that $(S \circ T)(\vec{x}) = \vec{0}$ has only the trivial solution. That is, we must show that if $(S \circ T)(\vec{x}) = \vec{0}$, then $\vec{x} = \vec{0}$. So assume that for some fixed $\vec{x}$ we have $(S \circ T)(\vec{x}) = S(T(\vec{x})) = \vec{0}$. Since S is one-to-one, this implies that $T(\vec{x}) = \vec{0}$. Since T is one-to-one, this implies that $\vec{x} = \vec{0}$, which was to be shown.

Approach 2: Another way to show that $S \circ T$ is one-to-one is to show that distinct inputs give distinct outputs. That is, we'll assume $S(T(\vec{x})) = S(T(\vec{y}))$ and show that $\vec{x} = \vec{y}$. Since S is one-to-one, our assumption implies that $T(\vec{x}) = T(\vec{y})$, as otherwise these two inputs of S would lead to the same output. Then because T is one-to-one, we have $\vec{x} = \vec{y}$, which was to be shown.