Name: ______________________________

Math 54, Summer 2009, Lecture 4
Quiz 4 Solutions

(1) Determine whether or not the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 2 & 2 & 1 \end{bmatrix}$ is invertible, and if it is find A^{-1}.

Setting up the matrix $[A : I]$ and row reducing, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 0 & -3 & 1 & -1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 1 & -1 & 0 \\ 0 & 1 & 0 & -8 & 1 & 4 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 7 & -1 & -3 \\ 0 & 1 & 0 & -8 & 1 & 4 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix}.$$

Thus A can be row reduced to I_3 (so its invertible) and

$$A^{-1} = \begin{bmatrix} 7 & -1 & -3 \\ -8 & 1 & 4 \\ 2 & 0 & -1 \end{bmatrix}.$$

(2) Say whether the statement is true or false. If it is true, justify your assertion. If it is false, provide a counterexample showing that it is false. (Problem continues on next page)

(a) If an $m \times n$ matrix A has a pivot in every row, then it is invertible.

False. The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has a pivot in every row, but is not square (so it certainly cannot be invertible). The statement is true if we also assume $m = n$.

(b) If A is 3×3 and there is exactly one vector $\vec{x}_0$ such that $A\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then A is invertible.

True. Since $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has a unique solution, A does not have any free variables. Thus A has a pivot in every column, and so by the invertible matrix theorem, A is invertible.

(3) If A is 6×4 and B is 4×6, show that the 6×6 matrix AB cannot be invertible. (Hint: think about the equation $B\vec{x} = \vec{0}$.)

Since B has more columns than rows, it must have a non-pivot column. Thus B has a free variable, so there is some $\vec{x}_0 \neq \vec{0}$ such that $B\vec{x}_0 = \vec{0}$. Thus $AB\vec{x}_0 = A\vec{0} = \vec{0}$, so $AB\vec{x} = \vec{0}$ has a non-trivial solution. By the invertible matrix theorem, AB is not invertible.

Note: we have a result that says that if A and B are square, then AB is invertible if and only if both A and B are invertible. This does not apply, as A and B are not square. We saw in class that it is possible for AB to be invertible when A and B are not both square. Recall:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and I_2 is certainly invertible. What goes wrong in the quiz problem is that the matrix with more columns than rows is on the right, not on the left.