Name: ______________________________

Math 54, Summer 2009, Lecture 4
Quiz 7 Solution

(1) For each of the following, either give an example of the matrix described or explain why such a matrix cannot exist.

(a) A matrix that has 1 as an eigenvalue but is not invertible.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) A 2×2 matrix with exactly 1 distinct eigenvalue.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) A 3×3 matrix whose eigenvalues are 2, 3, 4 and 5.

Cannot exist, since a 3×3 matrix has at most 3 eigenvalues.

(2) Suppose that V is a 2-dimensional vector space, and that $B = \{\vec{b}_1, \vec{b}_2\}$ and $C = \{\vec{c}_1, \vec{c}_2\}$ are two bases for this space. The basis vectors satisfy the relations $\vec{b}_1 = \vec{c}_1 + \vec{c}_2$ and $\vec{b}_2 = \vec{c}_1 + 2\vec{c}_2$. Find $[\vec{x}]_B$, where $\vec{x} = 2\vec{c}_1 + 3\vec{c}_2$.

By inspection, $[\vec{b}_1]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[\vec{b}_2]_C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \implies \underset{B \leftarrow C}{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Thus

$$[\vec{x}]_B = \underset{B \leftarrow C}{P}[\vec{x}]_C = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(3) Let $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a vector space V, and let $[\cdot]_B : V \to \mathbb{R}^n$ be the usual coordinate mapping. Prove that $[\cdot]_B$ is one-to-one.

We need to show that if $[\vec{u}]_B = [\vec{v}]_B$, then $\vec{u} = \vec{v}$. So assume $[\vec{u}]_B = [\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Then by the definition of coordinates

$$\vec{u} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n,$$

and

$$\vec{v} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n.$$

Thus $\vec{u} = \vec{v}$.