Math 54, Summer 2009, Lecture 4
Worksheet 1: Lay 1.7

(1) Classify the following sets as linearly independent or linearly dependent. (Hint: many don't require calculation).

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$.

(b) $\left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$.

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$.

(d) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(e) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 9 \\ -12 \end{bmatrix} \right\}$.

(a) Linearly dependent - contains $\vec{0}$.
(b) Linearly independent, because the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & -3 \\ 2 & 5 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{pmatrix}$$

has no free variables.
(c) Linearly dependent - there are more vectors than entries in the vectors. (d) Linearly dependent, because the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

contains a free variable.
(e) Linearly independent - there are only two vectors, and neither is a multiple of the other.

(2) True or False: The columns of a matrix A are linearly dependent if and only if the equation $A\vec{x} = \vec{0}$ is consistent. Justify your answer.

This statement is false. The equation $A\vec{x} = \vec{0}$ is always consistent (since it always has the solution $\vec{x} = \vec{0}$, at least). The correct statement is that the columns of A are linearly independent if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.

(3) Suppose $\vec{v}_1, \ldots, \vec{v}_4$ are vectors in $\mathbb{R}^3$. Let $S_2 = \{\vec{v}_1, \vec{v}_2\}$, $S_3 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and $S_4 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. For each of the following, mark the statement true or false. As always, justify your answer.

(a) If $\operatorname{Span} S_4 = \mathbb{R}^3$, then $\operatorname{Span} S_3 = \mathbb{R}^3$.

(b) If $\operatorname{Span} S_3 = \mathbb{R}^3$, then $\operatorname{Span} S_4 = \mathbb{R}^3$.

(c) If S_2 is linearly dependent, then so is S_3.

(d) If S_3 is linearly dependent, then so is S_2.

(a) False. Consider $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (2, 0, 0)$, $\vec{v}_3 = (0, 1, 0)$ and $\vec{v}_4 = (0, 0, 1)$. One can check that all four collectively span $\mathbb{R}^3$, but the first three do not.

(b) True. If $\vec{x} \in \operatorname{Span} S_3$, then there are coefficients c_j such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$. Then we have $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + 0\vec{v}_4$, so $\vec{x} \in \operatorname{Span} S_4$. Thus everything in $\operatorname{Span} S_3$ is also in $\operatorname{Span} S_4$. In particular, if $\operatorname{Span} S_3$ contains all of $\mathbb{R}^3$, then so does $\operatorname{Span} S_4$.

(c) True. If S_2 is linearly dependent, there are coefficients c_1, c_2, not both 0, such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Then $c_1\vec{v}_1 + c_2\vec{v}_2 + 0\vec{v}_3 = \vec{0}$, and not all of the coefficients are 0. Thus S_3 is linearly dependent.

(d) False. Consider $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (0, 1, 0)$, and $\vec{v}_3 = (0, 2, 0)$.

The moral of the story: when you add vectors to a set, the span can't get any smaller, but it has a chance of making a set linearly dependent.