Math 117: Homework 6
Due Tuesday, November 20th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Consider the sequences defined as follows:

\[ a_n = (-1)^{n+1}, \quad b_n = -\frac{1}{n}, \quad c_n = 2n, \quad d_n = \frac{3n+1}{4n-1}. \]

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, give its set of subsequential limits. Justify your answer.

(c) For each sequence, give its lim inf and lim sup. Justify your answer.

(d) Which of the sequences converges? Diverges to +\(\infty\)? Diverges to -\(\infty\)? Justify your answer.

(e) Which of the sequences is bounded? Justify your answer.

Question 2

Follow the same instructions as in the previous question for the following sequences:

\[ s_n = \cos\left(\frac{n\pi}{3}\right), \quad t_n = \frac{3}{4n+1}, \quad u_n = \left(-\frac{1}{2}\right)^n, \quad v_n = (-1)^n + \frac{1}{n}. \]

Question 3* (Similar to 11.10)

Let \( s_n \) be the sequence defined in the following figure from the textbook:

![Figure 11.2](Exercises 77)

11.5 Let \((q_n)\) be an enumeration of all the rationals in the interval \((0, 1]\).

(a) Give the set of subsequential limits for \((q_n)\).

(b) Give the values of \(\text{lim sup } q_n\) and \(\text{lim inf } q_n\).

11.6 Show every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence.

Hint: Define subsequence as in Definition 11.1.

11.7 Let \((r_n)\) be an enumeration of the set \(\mathbb{Q}\) of all rational numbers. Show there exists a subsequence \((r_{n_k})\) such that \(\lim_{k \to \infty} r_{n_k} = +\infty\).

11.8 \(*\) Use Definition 10.6 and Exercise 5.4 to prove \(\text{lim inf } s_n = -\text{lim sup } (-s_n)\) for sequence \((s_n)\).

11.9

(a) Show the closed interval \([a, b]\) is a closed set.

(b) Is there a sequence \((s_n)\) such that \((0, 1)\) is its set of subsequential limits?

8 This exercise is referred to in several places.
Question 4* (Similar to 12.12)

Let \( s_n \) be a sequence of nonnegative numbers, and for each \( n \) define \( \sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n) \).

(a) Show \( \lim \inf s_n \leq \lim \inf \sigma_n \leq \lim \sup \sigma_n \leq \lim \sup s_n \).

(\textbf{Hint:} For the first inequality, show that \( M > N \) implies
\[
\inf\{\sigma_n : n > M\} \geq \left(1 - \frac{N}{M}\right)\inf\{s_n : n > N\}.
\]
For the last inequality, show first that \( M > N \) implies
\[
\sup\{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.
\]

(b) Show that if \( \lim s_n \) exists, then \( \lim \sigma_n \) exists and \( \lim \sigma_n = \lim s_n \).

(c) Give an example for which \( \lim \sigma_n \) exists but \( \lim s_n \) does not exist.

Question 5 (Similar to 12.13)

Let \((s_n)\) be a bounded sequence of real numbers. Let \( A \) be the set of \( a \in \mathbb{R} \) such that \( \{n \in \mathbb{N} : s_n < a\} \) is finite, i.e. \( A \) is the set of real numbers \( a \) for which only finitely many \( s_n \) are less than \( a \). Let \( B \) be the set of \( b \in \mathbb{R} \) such that \( \{n \in \mathbb{N} : s_n > b\} \) is finite. Prove \( \sup A = \lim \inf s_n \) and \( \inf B = \lim \sup s_n \).
**Background on Infinite Series**

In calculus, you encountered infinite series of the form

\[ \sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots. \]

In fact, these are just limits of sequences. In particular, if we define the sequence

\[ s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \]

to be the sum of the first \( n \) terms of the series, then

\[ \sum_{k=1}^{\infty} a_k = \lim_{n \to +\infty} s_n. \]

**DEFINITION 1.** Given a series \( \sum_{k=1}^{\infty} a_k \), define the sequence \( s_n = \sum_{k=1}^{n} a_k \). Then the series \( \sum_{k=1}^{\infty} a_k \) converges to a number \( L \) if and only if the sequence \( s_n \) converges to \( L \). Likewise, the series diverges to \( +\infty \) or \( -\infty \) if and only if the sequence \( s_n \) diverges to \( +\infty \) or \( -\infty \).

**Question 6* (Cauchy criterion)**

Recall that a sequence \( (s_n) \) is a Cauchy sequence if and only if

\[ \text{for all } \epsilon > 0 \text{ there exists } N \in \mathbb{R} \text{ so that } n > m > N \text{ ensures } |s_n - s_m| < \epsilon. \]

(In our definition from class, we did not state \( n > m > N \), but instead \( n, m > N \), but we may assume \( n > m > N \) without loss of generality.)

(a) Prove the following theorem about series, known as the Cauchy criterion.

**THEOREM 1.** A series \( \sum_{k=1}^{\infty} a_k \) is convergent if and only if

\[ \text{for all } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ so that } n > m > N \text{ implies } \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon. \]

(b) Now use the theorem you proved in part (a) to prove the following corollary:

**COROLLARY 2.** If the series \( \sum_{k=1}^{\infty} a_k \) is convergent, then \( \lim_{k \to +\infty} a_k = 0 \).

(Hint: take \( n = m + 1 \) in the theorem from part (a).)

**Question 7 (Similar to 14.5)**

Suppose \( \sum_{k=1}^{\infty} a_k = A \) and \( \sum_{k=1}^{\infty} b_k = B \) for \( A, B \in \mathbb{R} \).

(a) Use the limit theorems for sequences to prove that \( \sum_{k=1}^{\infty} (a_k + b_k) = A + B \).

(b) Use the limit theorems for sequences to prove that for \( c \in \mathbb{R} \), \( \sum_{k=1}^{\infty} ca_k = cA \).
Question 8* (geometric series)

On HW4, Q3, and HW5, Q7, you proved the following results:

\[
\lim_{n \to +\infty} r^n = \begin{cases} 
0 & \text{if } |r| < 1 \\
1 & \text{if } |r| = 1 \\
+\infty & \text{if } r > 1 \\
does not exist & \text{if } r \leq -1,
\end{cases}
\]

and

\[
\text{for } r \neq 1, \quad \sum_{k=1}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.
\]

(a) Prove that for \( |r| < 1 \), \( \sum_{k=1}^{\infty} r^k = \frac{1}{1-r} \).

(b) Prove that for \( |r| > 1 \), \( \sum_{k=1}^{\infty} r^k \) does not converge. (Hint: Show that \( \lim_{k \to +\infty} r^k \neq 0 \) and use the corollary from Q6(c).)

Question 9* (absolute value of a series)

In general, the expression \( \sum_{k=1}^{\infty} a_k \) doesn’t always have meaning, since the limit of the corresponding sequence \( s_n = \sum_{k=1}^{n} a_k \) doesn’t always exist. On the other hand, in this problem you will show that the expression \( \sum_{k=1}^{\infty} |a_k| \) always has meaning.

(a) Prove that \( \sum_{k=1}^{\infty} |a_k| \) is either convergent or diverges to \( +\infty \). (Hint: Show that the corresponding sequence \( s_n = \sum_{k=1}^{n} |a_k| \) is monotone.)

(b) Prove that if \( \sum_{k=1}^{\infty} |a_k| \) is convergent, then \( \sum_{k=1}^{\infty} a_k \) is also convergent. (Hint: By HW5, Q7(c) you know that \( |\sum_{k=m}^{m} a_k| \leq \sum_{k=m}^{n} |a_k| \). Combine this fact with the theorem you proved in Q6(a).)