Recall:

**Def (sequence):** A sequence is a function whose domain is a set of the form \( \{m, m+1, m+2, \ldots \} \) for some \( m \in \mathbb{Z} \).

**Def (subsequence):** Consider a sequence \( s_n \).
A subsequence of \( s_n \) is a sequence of the form \( s_{n_k} \), where \( n_k \) is a sequence of integers satisfying \( n_1 < n_2 < n_3 < \ldots \).

Note: \( k \in \mathbb{N} = \{1, 2, 3, 4, \ldots \} \)

Informally, a subsequence is any infinite collection of elements from the original sequence, listed in order.

**Lemma:** Given a sequence \( s_n \), \( n \in \mathbb{N} \), if \( s_{n_k} \) is a subsequence, then \( n_k \geq k \) for all \( k \in \mathbb{N} \).

**Def (subsequential limit):** A subsequential limit of a sequence \( s_n \) is a real number or symbol \( +\infty \) or \( -\infty \) that is the limit of some subsequence of \( s_n \).
Thm: If a sequence $s_n$ converges to a limit $s$, then every subsequence also converges to $s$.

**Thm (main subsequences theorem)**

Let $s_n$ be a sequence of real numbers.

(a) Let $t \in \mathbb{R}$.

- $t$ is a subsequential limit of $s_n$ if and only if
  - the set $\mathcal{E}: |s_n - t| < \varepsilon$ is infinite for all $\varepsilon > 0$.

(b) If $s_n$ is unbounded above, $+\infty$ is a subsequential limit.

(c) If $s_n$ is unbounded below, $-\infty$ is a subsequential limit.

**Mental image (a):**

- Graph showing $s_n$'s behavior with subsequences indicated.
- Points $t$ indicating potential subsequential limits.
Why are subsequences important? We know a lot about monotone sequences, but "most" sequences aren't monotone.

**Thm:** Every sequence $s_n$ has a monotone subsequence.

Consequently, for any sequence $s_n$, we can always find a subsequence $s_{n_k}$ st. $\lim_{k \to \infty} s_{n_k}$ exists.
A simple corollary of this theorem...

**Theorem (Bolzano-Weierstrass):** Every bounded sequence has a convergent subsequence.

**Proof:** If \( s_n \) is a bounded sequence, then all subsequences are also bounded. Since every sequence has a monotone subsequence, there exists a subsequence \( s_{n_k} \) that is bounded and monotone. Thus, \( s_{n_k} \) is convergent.

Katy really likes \( \limsup \)'s and \( \liminf \)'s. Unfortunately, in general \( a_n \), \( b_n \) are not subsequences.

**Example:** Let \( s_n = n^2 \). Then \( a_N = \sup_{n > N^2} s_n = \infty \) for all \( N \).

Good news:

**Theorem:** For any sequence \( s_n \), \( \limsup s_n \) and \( \liminf s_n \) are subsequential limits.
Notation: $\bar{\mathbb{R}} = \{ -\infty \} \cup \mathbb{R} \cup \{ +\infty \}$

"extended real numbers"

Thm: Let $S \subseteq \bar{\mathbb{R}}$ denote the set of subsequential limits. Then $\limsup_{n \to \infty} s_n = \max(S)$ and $\liminf_{n \to \infty} s_n = \min(S)$.

Informally, $\limsup_{n \to \infty} s_n$ is the largest subsequential limit and $\liminf_{n \to \infty} s_n$ is the smallest subsequential limit.

Ex: $s_n = \{ -1, 0, 1, 0, -1, 0, 1, 0, \ldots, -\cos\left(\frac{n\pi}{2}\right), \ldots \}$

$S = \{ \text{subsequential limits} \} = \{ -1, 0, 1 \}$

$\liminf_{n \to \infty} s_n = -1$, $\limsup_{n \to \infty} s_n = 1$