Recall:

**Thm (sum, product, quotient cts fns):** If \( f \) and \( g \) cts at \( x_0 \in \text{dom}(f) \cap \text{dom}(g) \), then

(a) \( f + g \) is cts at \( x_0 \)
(b) \( fg \) is cts at \( x_0 \)
(c) \( \frac{f}{g} \) is cts at \( x_0 \), provided \( g(x_0) \neq 0 \)

**Thm (composition of cts fns is cts):** Suppose \( f \) is cts \( x_0 \) and \( g \) is cts at \( f(x_0) \). Then \( g \circ f \) is cts at \( x_0 \).

**Def:** (bounded function): \( f \) is bounded if there exists \( M > 0 \) s.t. \( |f(x)| \leq M \) for all \( x \in \text{dom}(f) \).

**Rmk:** \( f \) is a bounded function if and only if \( \{f(x) : x \in \text{dom}(f)\} \) is a bounded set.

\[ \text{range}(f) = \text{image}(f) \]

**Ex:**

- \( \text{dom}(f) = [1, \infty) \)
- \( f \) is cts
- \( f \) isn't bounded
- \( f \) is bdd on any \( [a, b] \subseteq \text{dom}(f) \)
This is true for all cts fn's.

**MAJOR RESULT**

**Thm (cts fn's attain max or min on closed intervals):**
A continuous function $f$ on a closed interval $[a, b] \subseteq \text{dom}(f)$ attains its maximum and minimum. In particular:

(i) $\text{max}$ and $\text{min}$ of $f$ on $[a, b]$ exist

(ii) $\exists \ x_{\text{max}}, \ x_{\text{min}} \in [a, b]$ so that

$$f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}}) \text{ for all } x \in [a, b].$$

The minimizer is the value of $x \in [a, b]$ so that

the minimum of $f$ on $[a, b]$ is $f(x)$.

A cts fn is only guaranteed to attain its max and min on a closed interval

Ex: $f(x) = \frac{1}{x^2}$ doesn't attain max, since $[a, b] \not\subseteq \text{dom}(f)$. 
Before we turn to proof: recall that if \( \frac{\sum_{n=1}^{N} a_n}{b_n} \) and \( \lim_{n \to \infty} x_n = x_0 \), then \( x_0 \in [a, b] \).

**Pf:**

### Step 1

Prove that \( f \) is bounded on \([a, b]\).

Assume, for the sake of contradiction that \( f \) is not bounded on \([a, b]\), that is for all \( M > 0 \), there exists \( x \in [a, b] \) so that \( |f(x)| > M \).

In particular, for all \( n \in \mathbb{N} \), there exists \( x_n \in [a, b] \) so that \( |f(x_n)| > n \). By Bolzano-Weierstrass, \( \exists x_{n_k} \) that is convergent.

Let \( x_0 = \lim_{k \to \infty} x_{n_k} \). We know \( x_0 \in [a, b] \).

Furthermore, since \( |f(x_{n_k})| > n_k \geq k \),

\[
\lim_{k \to \infty} |f(x_{n_k})| = +\infty.
\]

This contradicts the fact that, since \( f \) is acts \( fn \), \( \lim_{k \to \infty} |f(x_{n_k})| = |f(x_0)| \leq \infty \).

Therefore \( f \) is bounded on \([a, b]\).

### Step 2

Prove that \( f \) attains its max and min on \([a, b]\).

Since \( f \) is bounded on \([a, b]\), \( \exists M > 0 \) s.t. \( |f(x)| \leq M \) \( \forall x \in [a, b] \) \( \Rightarrow -M \leq f(x) \leq M \) \( \forall x \in [a, b] \).

Consequently, \( \sup \{ f(x) : x \in [a, b] \} = M_0 \) for \( M_0 \in \mathbb{R} \), since the \( 0 \) set is bdd above.
Since $M_0$ is the least upper bound, for all $n \in \mathbb{N}$, $M_0 - \frac{1}{n}$ is not an upper bound. Thus $\exists \ x_n \in [a,b]$ with $M_0 - \frac{1}{n} < f(x_n) \leq M_0$.

Since $x_n$ is bounded, by Bolzano–Weierstrass, it has a convergent subsequence $x_{n_k}$ with $\lim_{k \to \infty} x_{n_k} = x_0$ for $x_0 \in [a,b]$. Since $f$ is cts, $\lim_{k \to \infty} f(x_{n_k}) = f(x_0)$. By squeeze lemma and (4), $f(x_0) = M_0$. Thus, $f$ attains its max on $[a,b]$.

The proof that $f$ attains its min on $[a,b]$ is similar.

Now onto last major property of cts fn's...

**MAJOR RESULT**

**Thm (Intermediate Value Theorem):** If $f$ cts on an interval $I$, then for all $x, y \in I$, if $y$ lies between $f(a)$ and $f(b)$ then there exists $x$ between $a$ and $b$ such that $f(x) = y$. 

\[
\begin{array}{c}
\text{either } f(a) \leq y \leq f(b) \\
\text{or } f(b) \leq y \leq f(a)
\end{array}
\]
Ex:

Suppose $y$ is an intermediate value, that is $f(a) \leq y \leq f(b)$ for some $a, b \in I$. We want to show there exists a $\alpha \in [a, b]$ such that $f(\alpha) = y$. (We will consider the case $a \leq b$. The case $b \leq a$ is similar.)

Define

$$S = \{ x \in [a, b] : f(x) \leq y \}$$

Let $x_0 = \sup(S) \in [a, b]$. For all $n \in \mathbb{N}$, $x_0 - \frac{1}{n}$ is not an upper bound for $S$, so there exists $x_n \in S$ such that $x_0 - \frac{1}{n} \leq x_n \leq x_0$. Thus, the squeeze lemma ensures $\lim_{n \to \infty} x_n = x_0$. Furthermore, $f(x_n) \leq y$ for all $n \in \mathbb{N}$. 

[Diagram showing a function with an intermediate value $y$ and the squeeze lemma being applied]
Since $f$ is cts, $\lim_{x \to x_0^-} f(x) = f(x_0) \leq y$.

It remains to show $f(x_0) = y$.

Case 1: $x_0 = b \implies f(x_0) = f(b) \geq y$.

Case 2: $x_0 < b$

Define $t_n = \min \{ \frac{b}{2}, x_0 + \frac{1}{n} \}$. By defn, $\lim_{n \to \infty} t_n = x_0$. Since $x_0 = \sup(S)$ and $t_n > x_0$, $t_n \notin S$. Thus $f(t_n) > y \quad \forall n$. Since $f$ is cts, $\lim_{n \to \infty} f(t_n) = f(x_0) \geq y$.

Thus, $f(x_0) = y$, i.e. the fn attains its intermediate value. \hfill \Box

Recall:

**The ($\varepsilon$-$\delta$ characterization of cts):** Given $f$ and $x_0 \in \text{dom}(f)$, $f$ is cts at $x_0$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.
In general, the choice of $\delta$ depends on both $\varepsilon$ and $x_0$.

However, there are some functions for which $\delta$ only depends on $\varepsilon$ and not $x_0$.

**Def (uniformly cts):** Given a function $f$ and $S \subseteq \text{dom}(f)$, $f$ is uniformly cts on $S$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x,y \in S$ and $|x-y| < \delta$ imply $|f(x) - f(y)| < \varepsilon$.