Recall: Limit Theorems

Def (bounded sequence): A sequence \( s_n \) is bounded if there exists \( M \) s.t. \( |s_n| \leq M \) for all \( n \).

Ex: \( s_n = \cos(n\pi) \), since \( |s_n| \leq 1 \), it is a bounded sequence \( s_n = n \) is not bounded

Thm: Convergent sequences are bounded.

Thm (limit of sum is sum of limits): If \( s_n \) and \( t_n \) are convergent sequences, then \( \lim_{n \to \infty} (s_n + t_n) = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n \).

Thm (limit of product is product of limits): If \( s_n \) and \( t_n \) are convergent sequences, then \( \lim_{n \to \infty} s_n t_n = (\lim_{n \to \infty} s_n)(\lim_{n \to \infty} t_n) \).

Thm (limit of quotient is quotient of limits): If \( s_n \) and \( t_n \) are convergent sequences, \( s_n \neq 0 \) for all \( n \), and \( \lim_{n \to \infty} t_n \neq 0 \), then \( \lim_{n \to \infty} \left( \frac{t_n}{s_n} \right) = \frac{\lim_{n \to \infty} t_n}{\lim_{n \to \infty} s_n} \).

Pf: See book

Thm (basic examples):
(a) \( \lim_{n \to \infty} \left( \frac{1}{n} \right)^p = 0 \) if \( p > 0 \).
(b) \( \lim_{n \to \infty} a^n = 0 \) if \( |a| < 1 \)
(c) \( \lim_{n \to \infty} n^{1/n} = 1 \)
(d) \( \lim_{n \to \infty} a^{1/n} = 1 \) if \( a > 0 \)

**Pf:** see book

**Ex:** Does \( s_n = \frac{n^2 - 2}{n^2 + 2} \) converge? Yes, to zero.
Let's prove this!
Note that \( s_n = \frac{1}{n} - \frac{2}{n^2} \).

First, we have \( 1 + \frac{2}{n^2} > 0 \) for all \( n \) and, since the limit of the sum is the sum of the limits,
\[ \lim_{n \to \infty} 1 + \frac{2}{n^2} = \lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{2}{n^2} = 1 + 0 = 1 > 0. \]

Next, since the limit of the sum is the sum of the limits,
\[ \lim_{n \to \infty} \frac{1}{n} - \frac{2}{n^2} = \lim_{n \to \infty} \frac{1}{n} + \lim_{n \to \infty} -\frac{2}{n^2} = 0 + 0 = 0. \]

Finally, since limit of quotient is quotient of limits,
\[ \lim_{n \to \infty} s_n = \frac{0}{1} = 0. \]

**Ex:** What is the limit of \( s_n = n^2 \)?
**Def (diverges to $\pm \infty$):** A sequence $s_n$ diverges 
**to $+\infty$** if for all $M > 0$, there exists $N$ s.t. 
$n > N$ ensures $s_n > M$. We write \( \lim_{n \to \infty} s_n = +\infty \).

Likewise, a sequence $s_n$ diverges **to $-\infty$** if for all 
$M < 0$, there exists $N$ s.t. $n > N$ ensures $s_n < M$. 
We write \( \lim_{n \to \infty} s_n = -\infty \).

**Remark:**
- If $s_n$ diverges to $\pm \infty$, it does not converge.
- We will say that a sequence $s_n$ "has a limit" 
or "the limit of $s_n$ exists" if either:
  1. $s_n$ converges \( \iff \lim_{n \to \infty} s_n \in \mathbb{R} \)
  2. $s_n$ diverges to $\pm \infty$ \( \iff \lim_{n \to \infty} s_n = \pm \infty \)

**A few limit theorems for sequences that diverge to $\pm \infty$:**

**Thm:** Suppose \( \lim_{n \to \infty} s_n = +\infty \) and \( \lim_{n \to \infty} t_n > 0 \). 
Then \( \lim_{n \to \infty} s_n t_n = +\infty \).

**Pf:** Practice Midterm
Thm: Suppose $s_n > 0$ for all $n$. Then $\lim_{n \to \infty} s_n = +\infty$ if and only if $\lim_{n \to \infty} \frac{1}{s_n} = 0$.

Proof: First, assume $\lim_{n \to \infty} s_n = +\infty$. Fix $\epsilon > 0$. Note that $1 < s_n < 3 \iff \frac{1}{s_n} > 0 < \frac{1}{\epsilon} < s_n$. Since $\lim_{n \to \infty} s_n = +\infty$, there exists $N$ s.t. $n > N$ ensures $s_n > \frac{1}{\epsilon}$, hence $\frac{1}{s_n} - 0 < \epsilon$.

Next, assume $\lim_{n \to \infty} \frac{1}{s_n} = 0$. Fix $M > 0$. Note that $s_n > M \iff \frac{1}{s_n} < \frac{1}{M} \iff 1 < s_n < M$. Since $\lim_{n \to \infty} \frac{1}{s_n} = 0$, there exists $N$ s.t. $n > N$ ensures $\frac{1}{s_n} - 0 < \frac{1}{M}$, hence $s_n > M$.

Thm: If $\lim_{n \to \infty} s_n = +\infty$, then $\lim_{n \to \infty} (-s_n) = -\infty$.

Proof: Fix $M < 0$. Note that $(-s_n) < M \iff s_n > -M$. Since $\lim_{n \to \infty} s_n = +\infty$, there exists $N$ s.t. $n > N$ ensures $s_n > -M$, hence $(-s_n) < M$.

So far, we have studied convergent, divergent, and bounded sequences. There are two more important types of sequences: monotone and Cauchy sequences.
Def (increasing/decreasing/monotone sequence):
A sequence \( s_n \) is...
- increasing if \( s_n \leq s_{n+1} \ \forall \ n; \)
- decreasing if \( s_n \geq s_{n+1} \ \forall \ n; \)
- monotone if it either increasing or decreasing.

Ex: \( a_n = 1 - \left(\frac{1}{2}\right)^n \) is increasing,
\( b_n = \sqrt{n} \) "
\( c_n = (-1)^n \) is not monotonic.

Remark: If \( s_n \) is increasing, then \( s_n \leq s_m \)
whenever \( n \leq m. \)

\textit{Major Result #3}

Thm: All bounded, monotone sequences converge.

Mental image:

\[
\begin{align*}
\text{Sn} & \quad \text{S} = \exists s_n : n \in \mathbb{N}^3 \\
\text{M} & \\
\sup(S) & \\
\rightarrow \text{n}
\end{align*}
\]