Question 1

(a) Let \( f \) be a continuous function on \([0, +\infty)\). Prove that if \( f \) is uniformly continuous on \([k, +\infty)\) for some \( k \geq 0 \), then \( f \) is uniformly continuous on \([0, +\infty)\).

(b) Prove that \( f(x) = \frac{1}{x} \) is uniformly continuous on \([1, +\infty)\).

(c) Prove that \( f(x) = \frac{1}{x} \) is not uniformly continuous on \([0, +\infty)\).

(d) What aspect of part (a) fails for \( f(x) = \frac{1}{x} \)?

Question 2

Let \( f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \)

(a) Prove \( f \) is continuous on \( \mathbb{R} \). (You may assume \( g_1(x) = x \) and \( g_2(x) = \sin(x) \) are continuous on \( \mathbb{R} \).)

(b) Prove \( f \) is uniformly continuous on any bounded subset of \( \mathbb{R} \).

Question 3

Suppose \( f(x) \) is a continuous function on \([0, +\infty)\) satisfying \( 0 \leq f(x) \leq x \) for all \( x \in [0, +\infty) \).
Fix \( x_0 \in [0, +\infty) \) and define a sequence \( x_n \) recursively as follows: \( x_1 = f(x_0) \) and \( x_n = f(x_{n-1}) \).

(a) Prove that \( x_n \) converges to some \( y_0 \in [0, +\infty) \).

(b) Prove that \( y_0 \) is a fixed point of \( f \), i.e. \( f(y_0) = y_0 \).

Question 4

The golden ratio \( \varphi = \frac{1 + \sqrt{5}}{2} \) is the limit of the following continued fraction

\[
1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\]

This problem will guide you through the steps to prove this. Define a sequence \( s_n \) recursively as follows:
\( s_1 = 1 \), and, for \( n \geq 1 \), \( s_{n+1} = 1 + \frac{1}{s_n} \).

(a) Prove that the subsequence of even terms is decreasing and the subsequence of odd terms is increasing.

(b) Prove that the subsequence of even terms and the subsequence of odd terms converge to \( \varphi = \frac{1 + \sqrt{5}}{2} \).

(c) Use the following fact to prove \( s_n \) converges to \( \varphi = \frac{1 + \sqrt{5}}{2} \). (You will prove this fact in Question 6.)

\[
\lim_{n \to +\infty} s_n = s \iff \text{every subsequence } s_{nk} \text{ of } s_n \text{ has a further subsequence } s_{nk_l} \text{ satisfying } \lim_{l \to +\infty} s_{nk_l} = s.
\]
Question 5

(a) State the definition of what it means for a function \( f \) to be bounded on a set \( S \).
(b) Prove that a function \( f \) is bounded on a set \( S \) if and only if \( \{ f(x) : x \in S \} \) is bounded.
(c) Suppose \( f \) is bounded on a set \( S \) and the set \( S \) is also bounded. Define \( M = \sup \{ f(x) : x \in S \} \). Prove that there exists a convergent sequence \( s_k \in S \) so that \( \lim_{k \to +\infty} f(s_k) = M \).
(d) Give an example of a bounded function \( f \), a bounded set \( S \), and a sequence \( s_k \in S \) so that \( \lim_{k \to +\infty} f(s_k) = M = \sup \{ f(x) : x \in S \} \) but \( \lim_{k \to +\infty} s_k \not\in S \). Does \( \max \{ f(x) : x \in S \} \) exist?

Question 6

In this problem, we will consider sequences \( s_n \) satisfying the following property:

\[ \exists s \in \mathbb{R} \text{ s.t. every subsequence } s_{n_k} \text{ of } s_n \text{ has a further subsequence } s_{n_{k_l}} \text{ satisfying } \lim_{l \to +\infty} s_{n_{k_l}} = s. \quad (*) \]

(a) Prove that if \( \lim s_n = s \), then property \( (*) \) holds.

(b) Prove that if property \( (*) \) holds, then \( \lim s_n = s \). (Hint: Prove the contrapositive using the fact that, if a sequence does not converge to a limit \( s \), there exists \( \epsilon > 0 \) and a subsequence \( s_{n_k} \) so that \( |s_{n_k} - s| \geq \epsilon \) for all \( k \).

Question 7

Lightning Round!

(a) Consider the functions \( f(x) = \frac{1}{x-1} \) and \( g(x) = e^x \).

(i) What is \( \text{dom}(f \circ g) \)? Does there exist \( x_0 \in (0, +\infty) \) so that \( f \circ g(x_0) = \pi \)?

(ii) Let \( h = f g \). What is \( \text{dom}(h) \)? Is \( h \) bounded on the set \( (-1, 0) \)?

(b) Suppose \( a_n = (\sin \frac{n\pi}{2})^2 \) and \( b_n = (-1)^n \).

(i) What are the set of subsequential limits of \( a_n \)? \( b_n \)?

(ii) Does \( a_n + b_n \) converge? Does \( 2a_n + b_n \) converge?

(c) Let \( s_n \) be the sequence defined in the below figure from the textbook.

(i) What is the set of subsequential limits of \( s_n \)?

(ii) What are \( \lim \sup s_n \) and \( \lim \inf s_n \)?