Math 117: Practice Midterm 2

Question 1 (20 points)

(a) State the definition of a Cauchy sequence.

(b) State what it means for a sequence \( s_n \) to not be Cauchy by negating the definition from part (a).

(c) Prove the following fact: if a sequence is not Cauchy, then there exists \( \epsilon > 0 \) and a subsequence \( s_{n_k} \) so that \( |s_{n_{2k}} - s_{n_{2k+1}}| \geq \epsilon \) for all \( k \).

Question 2 (25 points)

Suppose \( s_n \) is a bounded sequence and define \( s = \sup\{s_n : n \in \mathbb{N}\} \).

(a) Define what it means for \( s_n \) to be a bounded sequence.

(b) Suppose \( s_n < s \) for all \( n \in \mathbb{N} \). Prove that, for all \( k \in \mathbb{N} \), the set \( B_k := \{s_n : s_n > s - \frac{1}{k}\} \) has infinitely many elements. (Hint: Prove the result by contradiction, assuming that for some \( k \in \mathbb{N} \), \( B_k \) has finitely many elements. Consider the cases where \( B_k \) has zero elements and \( B_k \) has a positive number of elements separately.)

(c) Suppose \( s_n < s \) for all \( n \in \mathbb{N} \). Use part (b) to show that there exists a subsequence of \( s_n \) converging to \( s \).

(d) Now suppose \( s_n = s \) for some \( n \in \mathbb{N} \). Give an example of such a sequence that doesn’t have a subsequence converging to \( s \).

Question 3 (15 points)

(a) State the definition of what it means for a sequence \( t_n \) to diverge to \(+\infty\).

(b) Now, state what it means for a sequence \( t_n \) to not diverge to \(+\infty\), by negating the above definition.

(c) State the Bolzano-Weierstrass Theorem.

(d) Suppose \( s_n \) has a subsequence \( s_{n_k} \) that is bounded. Show that this implies \( s_n \) has a convergent subsequence. (Hint: any subsequence of \( s_{n_k} \) is also a subsequence of \( s_n \).)

(e) Suppose that \( s_n \) has no convergent subsequences. Prove that \( \lim_{n \to +\infty} |s_n| = +\infty \). (Hint: prove the result by contradiction, by showing that if \( \lim_{n \to +\infty} |s_n| \neq +\infty \), then \( s_n \) has a bounded subsequence.)

Question 4 (20 points)

Lightning Round!

You do not need to show your work or justify your answers.

(a) Consider the sequence \( a_n = \frac{-2n^3 + 3}{2n^2 + 4} \).

(i) Does the limit exist? If so, what is it?

(ii) What is the set of subsequential limits?

(iii) What is the lim inf and lim sup?

(b) Consider the sequence \( b_n = (-1)^n + \frac{1}{n} \).
(i) Does the limit exist? If so, what is it?
(ii) What is the set of subsequential limits?
(iii) What is the lim inf and lim sup?

(c) (9 points) State whether the following statements are true or false. If it is true, prove it. If it is false, provide a counterexample.

(i) Suppose \( s_n \) has a subsequence \( s_{n_k} \) so that \( \lim_{k \to +\infty} s_{n_k} = +\infty \). Then \( \limsup_{n \to +\infty} s_n = +\infty \).
(ii) Suppose \( s_n \) has a subsequence \( s_{n_k} \) so that \( \lim_{k \to +\infty} s_{n_k} = 0 \). Then \( \limsup_{n \to +\infty} s_n = 0 \).