Homework 4

(a) see section notes from 5/13/21

(b) see section notes from 5/13/21

(c) want to show there exists a subsequence \( r_{n_k} \) s.t. \( \lim_{k \to \infty} r_{n_k} = +\infty \)

Fact: \( \mathbb{N} \subseteq \mathbb{Q} \)

Thus, for each \( m \in \mathbb{N} \), there exists \( n \in \mathbb{N} \) s.t. \( r_n = m \).

Since there exist \( q \in \mathbb{Q} \) s.t. \( q \geq 1 \), we may choose \( r_{n_1} = 1 \).

Since there exist \( q \in \mathbb{Q} \) s.t. \( q \geq 2 \), (justify…)
we may choose \( n_2 \geq n_1 \) with \( r_{n_2} \geq 2 \)

Recall from class, the "Main Subsequences Thm":
\( s_n \) is unbounded above \( \iff +\infty \) is a subseq. limit.

Thus, it suffices to show \( r_n \) is unbounded above.
Recall that \( r_n \) is unbounded above if there does not exist \( M > 0 \) s.t. \( r_n \leq M \) for all \( n \in \mathbb{N} \); that is, for all \( M > 0 \), there exists \( n \in \mathbb{N} \) s.t. \( r_n \geq M \).

Fix \( M > 0 \). By the Archimedean Property, there exists \( m \in \mathbb{N} \) s.t. \( m > M \). By definition of \( r_n \), there exists \( n \in \mathbb{N} \) s.t. \( r_n = M \geq m \). Since \( M > 0 \) was arbitrary, \( r_n \) is unbounded above.

\[
\mathbf{S} = (0, +\infty)
\]
HW 4, 5 (c)

(a) prove that $g(x) = kx$ is continuous

(b) prove that $f(x) = |x|$ is continuous

(c) prove that if $g$ is continuous at $x_0 \in \text{dom}(g)$, then $|g|$ is continuous at $x_0$

Since $g$ is continuous at $x_0$ and $f(x) = |x|$ is continuous (on $\text{dom}(f) = \mathbb{R}$) $f \circ g = |g|$ is continuous at $x_0$. 
(5) \( f(x) = |x|, \quad ||a - b|| \leq |a - b| \)

Proof: Fix \( x_0 \in \text{dom}(f) = \mathbb{R} \). Fix \( \varepsilon > 0 \).

Scratchwork:
\[ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < 3 \varepsilon \Leftrightarrow |x| - |x_0| < 3 \varepsilon \leq |x - x_0| \]

Practice Quiz 4

\( f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \)

(2) Fix \( a, b \in \mathbb{R}, \ a < b \). WTS \( \exists y \in \mathbb{R} \setminus \mathbb{Q} \) with \( a < y < b \).

\( \sqrt{2}, \pi, \ldots \in \mathbb{R} \setminus \mathbb{Q} \)

\( \sqrt{2} + 2 = r \), we think \( r \in \mathbb{R} \setminus \mathbb{Q} \)

Suppose \( \sqrt{2} + 2 \in \mathbb{Q} \), so \( \sqrt{2} + 2 = \frac{n}{m} \) for \( n, m \in \mathbb{Z}, \ m \neq 0 \). Then \( \sqrt{2} = \frac{n}{m} - 2 \).

Since \( \mathbb{Q} \) is a field and \( \frac{n}{m}, -2 \in \mathbb{Q}, \ \frac{n}{m} - 2 \in \mathbb{Q} \). This contradicts that \( \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \).
We know there exists \( q \in \mathbb{Q} \) s.t. \( a < q < b \).

\[
q^+ \frac{1}{e^{\pi}}
\]

\[
\frac{\pi}{M} < b - q \iff \frac{\pi}{b - q} < M
\]