HW 2, Question 1

Let $S$ and $T$ be nonempty subsets of $\mathbb{R}$ that are bounded below. Define

$$S + T = \{ s + t : s \in S, t \in T \}.$$ 

Prove $\inf(S) + \inf(T) = \inf(S+T)$.

Hint:
Step 1: Show that for all $t \in T$, $\inf(S+T) - t$ is a lower bound for $S$.

By defn of $S+T$ and the infimum, $\inf(S+T)$ is a lower bound for $S+T$, so $s + t = \inf(S+T) \iff s \geq \inf(S+T) - t$ for all $s \in S, t \in T$. Thus, for all $t \in T$, $\inf(S+T) - t$ is a lower bound for $S$.

Scratchwork:
By defn of lower bound, we aim to show that $s \geq \inf(S+T) - t$ for all $s \in S$.
Note that $s \geq \inf(S+T) - t \iff s + t \geq \inf(S+T)$.
Step 2: Show that $\inf(S+T)-\inf(S)$ is a lower bound for $T$.

By Step 1, for all $t \in T$, $\inf(S+T)-t$ is a lower bound for $S$. By definition, $\inf(S)$ is the greatest lower bound of $S$.

Thus, $\inf(S) = \inf(S+T)-t$.

$\iff t \leq \inf(S+T)-\inf(S)$ for all $t \in T$.

Scratchwork

Goal: WTS $t \geq \inf(S+T)-\inf(S)$

$\iff \inf(S) \geq \inf(S+T)-t$

by definition, $\inf(S)$ is the greatest lower bound.

by step 1, this is a lower bound of $S$.

Good trick:
If you ever want to prove $a = b$, it's sufficient to prove
1. $a \leq b$
2. $b \leq a$
Since \( \inf(S+T) - \inf(S) \) is a lower bound for \( T \) and \( \inf(T) \) is the greatest lower bound,

\[
\inf(T) \geq \inf(S+T) - \inf(S).
\]

\[
\implies \inf(S) + \inf(T) \geq \inf(S+T). \tag{\star}
\]

It remains to prove the opposite inequality. Since \( \inf(S) \) and \( \inf(T) \) are lower bounds for \( S \) and \( T \), for all \( s \in S \) and \( t \in T \),

\[
\inf(S) \leq s \quad \text{and} \quad \inf(T) \leq t \implies \inf(S) + \inf(T) \leq s + t.
\]

Thus, \( \inf(S) + \inf(T) \) is a lower bound for \( S+T \). Since \( \inf(S+T) \) is the greatest lower bound,

\[
\inf(S) + \inf(T) \leq \inf(S+T). \tag{\star\star\star}
\]

Thus, combining inequalities (\star) and (\star\star\star), we obtain

\[
\inf(S) + \inf(T) = \inf(S+T). \quad \square
\]

It suffices to show that \( \inf(S) + \inf(T) \) is a lower bound for \( S+T \).
\[ \inf(S) \leq s \quad \forall s \in S \]
\[ \inf(T) \leq t \quad \forall t \in T \]

Thus, \( \inf(S) + \inf(T) \leq s + t \quad \forall s \in S, \; t \in T \)

**Question 12**

Alternatively, you could use the Basic Examples Theorem from class to get this immediately.

(a) \( \lim_{n \to \infty} (-\frac{1}{2})^n = 0 \).

Fix \( \epsilon > 0 \). Let \( N = \frac{\log(\epsilon)}{\log(\frac{1}{2})} \). Then, \( n > N \) ensures

\[ n > \frac{\log(\epsilon)}{\log(\frac{1}{2})} \quad \iff \quad n \log(\frac{1}{2}) < \log(\epsilon) \quad \iff \quad \log((\frac{1}{2})^n) < \log(\epsilon) \]

\[ \iff \quad (\frac{1}{2})^n < \epsilon \quad \iff \quad |(-\frac{1}{2})^n - 0| < \epsilon \]

Since \( \epsilon > 0 \) was arbitrary, by the defn of convergence, we have \( \lim_{n \to \infty} (-\frac{1}{2})^n = 0 \).

**Scratchwork:**

\[ |(-\frac{1}{2})^n - 0| < \epsilon \quad \iff \quad |(-\frac{1}{2})^n| < \epsilon \quad \iff \quad (\frac{1}{2})^n < \epsilon \]

\[ \iff \quad \log((\frac{1}{2})^n) < \log(\epsilon) \quad \iff \quad n \log(\frac{1}{2}) < \log(\epsilon) \]

\[ \iff \quad n > \frac{\log(\epsilon)}{\log(\frac{1}{2})} \]

\[ e^0 = 1 \quad 10^0 = 1 \]
\[
\frac{\log_a(b)}{\log_a(c)} = \frac{\ln(b)}{\ln(c)} = \frac{\log_c(b)}{\log_c(c)}
\]

\[
c^{\log_a(b)} = b, \quad c^{\ln(b)/\ln(c)} = (e^{\ln(c)})^{\ln(b)/\ln(c)} = e^{\frac{\ln(c)\ln(b)}{\ln(c)}} = b
\]

\[
\left(\alpha^{\log_a(c)}\right)^{\log_a(b)/\log_a(c)} = (\alpha^{\log_a(c)})^{\frac{\log_a(b)}{\log_a(c)}} = \alpha^{\frac{\log_a(b)}{\log_a(c)} \log_a(c)} = \alpha^{\log_a(b)} = b
\]

**b) \( \lim_{n \to \infty} \frac{1}{n^{5/6}} = 0 \)**

Fix \( \varepsilon > 0 \). Let \( N = \frac{1}{\varepsilon^{6/5}} \). Then \( n > N \) ensures

\[
\frac{1}{\varepsilon^{6/5}} < n \iff \frac{1}{\varepsilon} < n^{5/6} \iff \frac{1}{n^{5/6}} < \varepsilon \iff |\frac{1}{n^{5/6}} - 0| < \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, by definition of convergence, \( \lim_{n \to \infty} \frac{1}{n^{5/6}} = 0 \). \( \Box \)

**Scratchwork:**

\[
|\frac{1}{n^{5/6}} - 0| < \varepsilon \iff \frac{1}{n^{5/6}} < \varepsilon \iff \frac{1}{\varepsilon} < n^{5/6} \iff \frac{1}{\varepsilon^{6/5}} < n
\]

\( n \in \mathbb{N} \), unless otherwise specified.

**c) \( \lim_{n \to \infty} \frac{1}{n} \cos(n) = 0 \)**

Fix \( \varepsilon > 0 \). Let \( N = \frac{1}{\varepsilon} \). Then \( n > N \) ensures

\[
n > \frac{1}{\varepsilon} \iff \frac{1}{n} < \varepsilon \iff \frac{1}{n} |\cos(n)| < \varepsilon \iff |\frac{1}{n} \cos(n)| - 0| < \varepsilon.
\]

**Scratchwork:**

\[
0 \leq |\cos(n)| \leq 1
\]

\[
|\frac{1}{n} \cos(n)| - 0| < \varepsilon \iff \frac{1}{n} |\cos(n)| < \varepsilon \iff \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}
\]
Recall: Lec 4c, floor function, for $N \geq 1$, 
$n_{\text{LN}} = \max\{n \in \mathbb{N} : n \leq \sqrt{N}\}

\lfloor 10.5 \rfloor = 10