Common mistakes regarding density of $\mathbb{Q}$ in $\mathbb{R}$

$A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$

Claim: $\text{sup}(A) = 1$

Proof of Claim:

Since $1 - \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$, 1 is an upper bound for $A$.

Suppose $a \in \mathbb{R}$ is an upper bound for $A$. WTS $a \geq 1$. Assume for the sake of contradiction $a < 1$.

Common mistake: by density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ s.t. $a < q < 1$. Since $q \in A$, this contradicts that $a$ is an upper bound.

Scratchwork

$1 - \frac{1}{N_0} > a \iff 1 - a > \frac{1}{N_0} \iff N_0 > \frac{1}{1 - a}$

by the Archimedean Property

Since $a < 1$, $\frac{1}{1 - a} \in \mathbb{R}$, so there exists $N_0 \in \mathbb{N}$ s.t. $N_0 > \frac{1}{1 - a} \iff 1 - \frac{1}{N_0} > a$. This
contradicts that \( a \) was an upper bound for \( A \).

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HW 2, Q8
\[ A = \{ q \in \mathbb{Q} : a < q^2 \} \]

Claim: \( \inf A = a \)

Pf:
• By defn of \( A \), \( a \) is a lower bound for \( A \).
• Suppose \( a \) is not the greatest lower bound for \( A \), that is, there exists \( q_0 \) s.t. \( q_0 \) is a lower bound for \( A \) and \( q_0 > a \).
• Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), \( \exists x \in \mathbb{Q} \) s.t. \( a < x < q_0 \).
• Thus \( x \in A \).
• Thus \( q_0 \) is not a lower bound for \( A \).
• Hence \( a \) is the least upper bound for \( A \).
Practice Quiz 2, Q8

Consider $s_n$ s.t. $s_n \neq 0 \forall n$ for which
$L = \lim_{n \to \infty} \left| \frac{s_{n+1}}{s_n} \right|$ exists.

(a) WTS if $L < 1$, show that $\lim_{n \to \infty} s_n = 0$.

Pg 8

Step 1: Let $a = \frac{L+1}{2}$. Then $2 < a < 1$.

Step 2: WTS $\exists N$ s.t. $n > N$ ensures
$|s_{n+1}| < a |s_n|$.

Scratchwork:
$|s_{n+1}| < a |s_n| \iff |\frac{s_{n+1}}{s_n}| < a \text{ for } n > N$.
We know $\lim_{n \to \infty} \left| \frac{s_{n+1}}{s_n} \right| = L < a < 1$

Let $\varepsilon = a - L$. Since $\left| \frac{s_{n+1}}{s_n} \right|$ converges to $L$, there exists $N$ s.t. $n > N$ ensures
Thus, for \( n > N \), \( |s_{n+1}| < a |s_n| \).

**Step 3:** \( \text{WTS } |s_n| \leq a^{n-N-1} |s_{n+1}| \text{ for all } n > N \)

\[ = a^n a^{-N-1} |s_{n+1}| \]

**Base case:** \( n = N+1 \)

\( \text{WTS } |s_{N+1}| \leq a^{(N+1)-N-1} |s_{N+1}| \)

This holds since \( a^{(N+1)-N-1} = a^0 = 1 \).

**Inductive step:**

**Inductive hypothesis:**
Fix \( m \in \mathbb{N} \) and assume \( |s_{N+m}| \leq a^{(N+m)-N-1} |s_{N+1}| \).

\( \text{WTS } |s_{N+m+1}| \leq a^{(N+m+1)-N-1} |s_{N+1}| \).

By Step 2,
\[ |s_{N+m+1}| < a |s_{N+m}| \leq a a^{(N+m)-N-1} |s_{N+1}| \]
\[ = a^{(N+m+1)-N-1} |s_{N+1}| \]

**Step 4:** \( \text{WTS } \lim_{n \to \infty} s_n = 0 \).

Fix \( \varepsilon > 0 \). Define \( \tilde{\varepsilon} = \varepsilon a^{N+1} / |s_{N+1}| \).

By Q4, \( \lim_{n \to \infty} a^n = 0 \). Thus there exists
\[ n \text{ so that } n > N \text{ ensures} \]
\[ |a^n - 0| < \epsilon \iff |a^n| < \epsilon \iff |a^n| < \frac{\epsilon}{|s_{n+1}|} \]

By Step 3, \(|s_n| < \epsilon \iff |s_n - 0| < \epsilon\). Since \(\epsilon > 0\) was arbitrary, this shows \(\lim_{n \to \infty} s_n = 0\).

Scratchwork:
\[ |s_n| < \epsilon \iff |s_n - 0| < \epsilon \iff |a^n| < \frac{\epsilon}{|s_{n+1}|} \]

For \(\epsilon = \frac{\epsilon}{|s_{n+1}|} = \frac{\epsilon a^{n+1}}{|s_{n+1}|}\).

\[ \text{---} \]

If \(|a| < 1\), WTS \(\lim_{n \to \infty} a^n = 0\).

Scratchwork:
\[ |a^n - 0| < \epsilon \iff |a^n| < \epsilon \iff |a|^n < \epsilon \]

If \(a \leq -1\), WTS \(\lim_{n \to \infty} a^n\) does not exist.

Assume \(\lim_{n \to \infty} a^n\) does exist.
Case 1: \( \lim_{n \to \infty} a^n = s \) for \( s \in \mathbb{R} \)

Scratchwork:
\[ |a^n - s| < \varepsilon \iff s - \varepsilon < a^n < s + \varepsilon \]

Special case \( a = -1 \). Take \( \varepsilon = \frac{1}{2} \). By defn of convergence, \( \exists N \) s.t. \( n > N \) ensures
\[ |a^n - s| < \frac{1}{2} \iff s - \frac{1}{2} < a^n < s + \frac{1}{2} \]. For \( n \) even, this implies \( 1 < s + \frac{1}{2} \iff \frac{1}{2} < s \). For \( n \) odd, this implies \( s - \frac{1}{2} < -1 \iff s < -\frac{1}{2} \). This is a contradiction.

Case 2...

\[ s_n \neq 0 \ \forall n \in \mathbb{N} \]
\[ \lim_{n \to \infty} s_n = 0 \]
WTS \( \inf \{ |s_n| : n \in \mathbb{N} \} > 0 \)

Let \( s = \lim_{n \to \infty} s_n \).
Fix $\varepsilon = \frac{|s|}{2}$. Show $\exists N \text{ s.t. } n > N$ ensures $|s_n - s| < \varepsilon \Rightarrow |s_n| > \frac{|s|}{2}$.

Define $a = \min \{ |s_1|, |s_2|, |s_3|, \ldots, |s_N|, \frac{|s|}{2} \}$.

Show $a > 0$ and $a$ is a lower bound for $\|s_n\|: n \in \mathbb{N}^3$.

6. \[
\left| \frac{1}{s_n} - \frac{1}{s} \right| < 3 \\
\overset{1}{\Rightarrow} \left| \frac{s - s_n}{s_n s} \right| < 3 \\
\overset{2}{\Rightarrow} \left| s - s_n \right| < 3 \inf \{ |s_n|: n \in \mathbb{N}^3 \} \]

\[
\left| s - s_n \right| < 3 |s| \inf \{ |s_n|: n \in \mathbb{N}^3 \} 
\]