Practice Quiz 5

1) Suppose $f(x)$ is a continuous function on $[0, +\infty)$ satisfying $0 \leq f(x) \leq x$ for all $x \in [0, +\infty)$. Fix $x_0 \in [0, +\infty)$ and define a sequence $x_n$ recursively

$$x_n = f(x_{n-1}).$$

(a) WTS $x_n$ converges to some $y_0 \in [0, +\infty)$. Since $f(x) \leq x$, we have

$$x_n = f(x_{n-1}) \leq x_{n-1}$$

for all $n \in \mathbb{N}$. Since $0 \leq f(x)$, we have $x_n \geq 0$ for all $n \in \mathbb{N}$. Since $x_n$ is a bounded decreasing sequence and all bounded, monotone sequences converge, there exists $y_0$ so that

$$\lim_{n \to \infty} x_n = y_0.$$

Since $x_n \geq 0$ for all $n \in \mathbb{N}$, $y_0 \geq 0$.

(b) WTS $f(y_0) = y_0$.

We have $\lim_{n \to \infty} x_n = y_0$.

Since $f$ is continuous, $\lim_{n \to \infty} f(x_n) = f(y_0)$. 
Since \(\lim_{n \to \infty} x_n = y_0\), it suffices to show
\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n.
\]

Since \(f(x_n) = x_{n+1}\), it suffices to show
\[
\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n.
\]

You proved this was true on HW2, Q14.

\(\Box\)

Let \(f_n(x) = (x - \frac{1}{n})^2\) for \(x \in [0, 1]\).

(a) Does \(f_n\) converge pointwise on \([0, 1]?)

**Def (pointwise convergence):** Consider a sequence of functions \(f_n\), with \(S \subseteq \text{dom}(f_n)\) for all \(n \in \mathbb{N}\).

We say \(f_n\) **converges pointwise** to \(f\) on \(S\) if
\[
\lim_{n \to \infty} f_n(x) = f(x)\quad \text{for all } x \in S.
\]

"Converges for each point"

We will write \(\lim_{n \to \infty} f_n = f\) or \(f_n \to f\) pointwise.
Let $f(x) = x^2$. Fix $x \in S$. In homework 4, you proved that any polynomial is continuous on $\mathbb{R}$. Consider the polynomial $g(y) = (x - y)^2$. Since $\lim_{n \to \infty} \frac{1}{n} = 0$,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} (x - \frac{1}{n})^2 = \lim_{n \to \infty} g(\frac{1}{n}) = g(0) = x^2 = f(x).$$

Since $x \in S$ was arbitrary, this shows that $\lim_{n \to \infty} f_n(x) = f(x)$.

(b) Does $f_n$ converge uniformly on $[0, 1]$?

\[\begin{array}{c}
\text{Graphs of } f_n(x) = \frac{x^2}{n} \text{ for } n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. \\
\end{array}\]
Def (uniform convergence): Consider a sequence of functions \( f_n \), with \( S \subseteq \text{dom}(f) \) for all \( n \in \mathbb{N} \).

We say \( f_n \) converges uniformly to \( f \) on \( S \) if, for all \( \varepsilon > 0 \), there exists \( N \) such that \( n > N \) ensures \( |f_n(x) - f(x)| < \varepsilon \) for all \( x \in S \).

We will write \( \lim_{n \to \infty} f_n = f \) or \( f_n \to f \) uniformly.

Let \( f(x) = x^2 \). Let \( \varepsilon > 0 \).

Scratchwork:

\[
|f_n(x) - f(x)| < \varepsilon \\
|\left(x - \frac{1}{n}\right)^2 - x^2| < \varepsilon \\
\]

\( x \in [0, 1] \)

\[
|\left(x - \frac{1}{n}\right)^2 - x^2| = \left| -\frac{2x}{n} + \frac{1}{n^2}\right| \leq \frac{2x}{n} + \frac{1}{n^2} \\
\leq \frac{2x}{n} + \frac{1}{n^2} \leq \frac{2}{n} + \frac{1}{n^2} \leq \frac{2}{n} + \frac{1}{n} \\
= \frac{3}{n} \\
\]

Then \( \frac{3}{n} < \varepsilon \) if \( n > \frac{3}{\varepsilon} \).
\[ \text{let } N = \frac{3}{2}. \]