Homework 2 Solutions
(c) Kately Craig, 2021

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Step 1: Show that for all \( t \in T \), \( \inf(S+T) - t \) is a lower bound for \( S \).

By defn of \( S+T \) and the infimum, \( \inf(S+T) \) is a lower bound for \( S+T \), so \( s + t \leq \inf(S+T) \iff s \geq \inf(S+T) - t \) for all \( s \in S \), \( t \in T \). Thus, for all \( t \in T \), \( \inf(S+T) - t \) is a lower bound for \( S \).

Step 2: Show that \( \inf(S+T) - \inf(S) \) is a lower bound for \( T \).

By Step 1, for all \( t \in T \), \( \inf(S+T) - t \) is a lower bound for \( S \). By defn, \( \inf(S) \) is the greatest lower bound of \( S \). Thus, \( \inf(S) \geq \inf(S+T) - t \iff t \geq \inf(S+T) - \inf(S) \) for all \( t \in T \).
Since \( \inf(S+T) - \inf(S) \) is a lower bound for \( T \) and \( \inf(T) \) is the greatest lower bound,

\[
\inf(T) \geq \inf(S+T) - \inf(S).
\]

\[
\Rightarrow
\]

\[
\inf(S) + \inf(T) \geq \inf(S+T). \tag{\ast}
\]

It remains to prove the opposite inequality. Since \( \inf(S) \) and \( \inf(T) \) are lower bounds for \( S \) and \( T \), for all \( s \in S \) and \( t \in T \),

\[
\inf(S) \leq s \quad \text{and} \quad \inf(T) \leq t \Rightarrow \inf(S) + \inf(T) \leq s + t.
\]

Thus, \( \inf(S) + \inf(T) \) is a lower bound for \( S+T \). Since \( \inf(S+T) \) is the greatest lower bound,

\[
\inf(S) + \inf(T) \leq \inf(S+T). \tag{\ast\ast\ast}
\]

Thus, combining inequalities (\ast) and (\ast\ast\ast), we obtain

\[
\inf(S) + \inf(T) = \inf(S+T). \quad \Box
\]
slight notational change: \( A = S, B = T \)

2. a) Since \( s \leq t \) for all \( s \in S \) and \( t \in T \), any \( t \in T \) is an upper bound for \( S \) and any \( s \in S \) is a lower bound for \( T \). Hence, \( S \) is bounded above and \( T \) is bounded below.

b) As shown in part(a), any \( t \in T \) is an upper bound for \( S \). Since \( \sup(S) \) is the least upper bound, \( \sup(S) \leq t \) for all \( t \in T \). Thus, \( \sup(S) \) is a lower bound for \( T \), and since \( \inf(T) \) is the greatest lower bound, \( \sup(S) \leq \inf(T) \).

c) \( S = [0,1] \), \( T = [1,2] \)
d) \( S = [0,1) \), \( T = (1,2] \)

3. Throughout, we use \( S \) to denote the set under consideration.
   a) \( \sup(S) = \sqrt{2}, \inf(S) = -\sqrt{2} \)
   b) \( \sup(S) = \pi, \inf(S) = -1 \)
   c) \( \sup(S) = \inf(S) = 1 \)
   d) \( S \) is not bounded above, \( \inf(S) = 1 \)
   e) \( \sup(S) = 1, \inf(S) = 0 \)
   f) \( \sup(S) = 1, \inf(S) = -1 \)
   g) \( S = [-1,1], \) so \( \sup(S) = 1 \) and \( \inf(S) = -1 \)
4. \(\sup(S) = 1, \inf(S) = 0\)
   a. \(S\) is not bounded above, \(\inf(S) = 0\)
   b. \(S\) is not bounded above, \(\inf(S) = 0\)
   c. \(S\) is not bounded above, \(\inf(S) = 0\)
   d. \(S\) is neither bounded above or below
   e. \(S = \{0^3\}, \sup(S) = \inf(S) = 0\)
   f. \(S\) is not bounded above, \(\inf(S) = 2^{1/3}\)
   g. \(\sup(S) = \inf(S) = 0\)

5. Let \(S = (a, b]\).
   - \(\max(S) = b\), since by defn., \(b\) is the largest element in \(S\)
   - \(\sup(S) = b\). \(b\) is an upper bound for \(S\) and since \(b \in S\), no number smaller than \(b\) can be an upper bound. Thus \(b\) is the least upper bound.
   - The minimum of \(S\) does not exist. Suppose, for the sake of contradiction that \(\min(S) = m_0\).
     Since \(m_0 \in S\), \(m_0 > a\). However \(\frac{m_0 + a}{2} \in (a, m_0)\), so \(\frac{m_0 + a}{2} \in S\) and \(\frac{m_0 + a}{2} < m_0\). This contradicts that \(m_0\) was the smallest element in \(S\).
   - \(\inf(S) = a\). \(a\) is a lower bound for \(S\). Suppose \(m_0 > a\) was another lower bound. Since \(b \in S\), we have \(m_0 \in (a, b]\). Furthermore, since \(\frac{m_0 + a}{2} \in (a, m_0)\), we have \(\frac{m_0 + a}{2} \in S\) and \(\frac{m_0 + a}{2} < m_0\).
This contradicts that \( m_0 \) was a lower bound of \( S \).

\[ \text{slight change: let } x = -a \]

6. By the Archimedean Property, if \( x > 0 \) and \( y > 0 \), then there exists \( n \in \mathbb{N} \) so that \( nx > y \).

Taking \( x = 1 \) and \( y = a \) gives that there exists \( n_1 \in \mathbb{N} \) so that \( n_1 > a \).

Taking \( x = a \) and \( y = 1 \) gives that there exists \( n_2 \in \mathbb{N} \) so that \( n_2 a > 1 \Rightarrow a > \frac{1}{n_2} \).

Let \( n = \max \{ n_1, n_2 \} \). Then \( n \in \mathbb{N} \) and \( \frac{1}{n} \leq \frac{1}{n_2} < a < n_1 < n \), which gives the result.

7. Suppose for the sake of contradiction that \( a > b \). Then if we define \( y = a - b, y > 0, \) and by Q9, there exists \( n \in \mathbb{N} \) so that \( \frac{1}{n} < y = a - b \). This implies that there exists \( n \in \mathbb{N} \) so that \( b + \frac{1}{n} < a \), which is a contradiction. Therefore, we must have \( a \leq b \).
Define \( S = \{ q \in \mathbb{Q} : a < q^2 \} \). By definition, \( a \) is a lower bound for \( S \). Assume for the sake of contradiction that there exists another lower bound \( m_0 \) of \( S \) such that \( a < m_0 \). By denseness of \( \mathbb{Q} \) in \( \mathbb{R} \), there exists \( r \in \mathbb{Q} \) s.t. \( a < r < m_0 \). Then \( r \in S \), which contradicts the fact that \( m_0 \) was a lower bound of \( S \). Therefore for all lower bounds \( m_0 \) of \( S \), we must have \( m_0 \leq a \). This shows that \( a \) is the greatest lower bound of \( S \), i.e. \( \inf(S) = a \).

First assume that \( S \) is bounded below by some \( a > 0 \). Then by Corollary 1, \( S \) has an infimum, and since \( \inf(S) \) is the greatest lower bound, we have \( 0 \leq a \leq \inf(S) \).

Using that \( \inf(S) \) is a lower bound for \( S \), we have \( \inf(S) \leq s \) \( \forall s \in S \). Since \( \inf(S) > 0 \), this is equivalent to \( \frac{1}{\inf(S)} \leq \frac{1}{s} \) \( \forall s \in S \).

Thus \( \frac{1}{\inf(S)} \) is an upper bound for \( S \).
Suppose $M_0$ is also an upper bound for $S'$, so $\frac{1}{s} \leq M_0 \ \forall s \in S$. This implies $\frac{1}{M_0} \leq s \ \forall s \in S$, so $\frac{1}{M_0}$ is a lower bound for $S$. By definition, $\inf(S)$ is the greatest lower bound for $S$, so $\frac{1}{M_0} \leq \inf(S) \Rightarrow \inf(S) \leq M_0$. This shows that $\inf(S)$ is smaller than any other upper bound for $S'$. Therefore $\inf(S) = \sup(S)$.

Finally, suppose that $S$ is not bounded below by any $a > 0$. Suppose for the sake of contradiction that $S'$ is bounded above by some $M_0$. Then $\frac{1}{s} \leq M_0 \ \forall s \in S$. Since $S = \{x \in \mathbb{R} : x > 0\}$, we must have $M_0 > 0$. Therefore $\frac{1}{M_0} \leq s \ \forall s \in S$. This contradicts that $S$ is not bounded below by any $a > 0$. Therefore $S'$ must not be bounded above, i.e. $\sup(S') = +\infty$.

Case 1: Suppose $S$ is not bounded below. Then $\inf(S) = -\infty$ and $-\infty$ is less than or equal to any real number, so $\inf(S) \leq \sup(S)$. 

(slight notational change: $A = S$)
Case 2: Suppose $S$ is not bold above. Then $\sup(S) = +\infty$ and $+\infty$ is greater than or equal to any real number or $-\infty$, so $\inf(S) \leq \sup(S)$.

Case 3: Suppose $S$ is bold. Since $S$ is nonempty, $\inf(S)$ is a lower bound for $S$, and $\sup(S)$ is an upper bound for $S$, we have $\inf(S) \leq s \leq \sup(S)$, $\forall s \in S$.

11  (a) Define $x_n = \frac{\sqrt{2}}{n}$. As shown in class, $\sqrt{2}$ is an irrational number. Since $\mathbb{Q}$ is a field, the product of two rational numbers is a rational number. Since $1 \in \mathbb{Q}$ and $x_n \cdot n = \sqrt{2} \in \mathbb{Q}$, we must have that $x_n \in \mathbb{Q}$, so $\{x_n\}$ is a sequence of irrational numbers.

Claim: $\lim_{n \to \infty} x_n = 0$. We must show that for all $\varepsilon > 0$, there exists $N$ s.t. $n > N$ ensures $|x_n| < \varepsilon$. Note that $|x_n| = \left|\frac{\sqrt{2}}{n}\right| < \frac{\sqrt{2}}{n} < \varepsilon \iff \frac{\sqrt{2}}{\varepsilon} < n$.

Therefore, for all $\varepsilon > 0$, if we take $N = \frac{\sqrt{2}}{\varepsilon}$, then for all $n > N$, $|x_n| < \varepsilon$. 
(b) Define \( r_n = 1.41421 \ldots \) as the first \( n \) digits of the decimal approximation of \( \sqrt{2} \).

Or more precisely, we define \( r_n \) by \( r_n = \left\lfloor \frac{\sqrt{2}}{10^n} \right\rfloor / 10^n \), where \( \left\lfloor a \right\rfloor \) represents the largest integer less than or equal to \( a \). Then \( r_n \in \mathbb{Q} \).

Claim: \( \lim_{n \to \infty} r_n = \sqrt{2} \). Note that
\[
|r_n - \sqrt{2}| = 10^{-n} |\sqrt{2} - 1| / 10^n = 10^{-n} |\sqrt{2} - 1| - 10^{-n} \leq 10^{-n}
\]
and
\[
10^{-n} < \epsilon \iff 10^{-n} \frac{1}{\epsilon} < 10^{n} \iff \log_{10} \frac{1}{\epsilon} < n.
\]

Therefore, for all \( \epsilon > 0 \), if we take \( N = \log_{10} \frac{1}{\epsilon} \), then for all \( n > N \),
\[
|r_n - \sqrt{2}| < \epsilon.
\]

(12) Note that \( |1 - \frac{1}{2}^n| = \left| \frac{1}{2} \right|^n < \frac{1}{2^n} \iff \log_{2} \frac{1}{2} < n \). Therefore, for all \( \epsilon > 0 \), if we take \( N = \log_{2} \frac{1}{\epsilon} \), then \( n > N \) ensures
\[
|1 - \frac{1}{2}^n| < \epsilon.
\]

(b) Note that \( |\frac{1}{n^{5/6}} - 0| = \frac{1}{n^{5/6}} < \epsilon \iff \frac{1}{\epsilon} < n^{5/6} \iff \left( \frac{1}{\epsilon} \right)^{6/5} < n \). Therefore, for all \( \epsilon > 0 \), if we take \( N = \left( \frac{1}{\epsilon} \right)^{6/5} \), then \( n > N \) ensures
\[
|\frac{1}{n^{5/6}} - 0| < \epsilon.
\]
(c) Note that 
\[
\frac{|5n+2 - \frac{5}{2}|}{2n+2} = \frac{|10n+4 - 10n - 10|}{4n+4} = \frac{6}{4n+4} = \frac{3}{2n+2} \leq \frac{4}{2n+2} \leq \frac{2}{2n} = \frac{1}{n} \text{ and } \frac{2}{n} < 3 \Leftrightarrow \frac{2}{3} < n.
\]
Therefore, for all \( \varepsilon > 0 \), if we take \( N = \frac{2}{\varepsilon} \), then \( n > N \) ensures \( \frac{|5n+2 - \frac{5}{2}|}{2n+2} < \varepsilon \).

(d) Note that 
\[
\frac{|n-1|}{n^2-1} - 0 = \frac{|n-1|}{n^2-1} = \frac{1}{n+1} \leq \frac{1}{n}
\]
and \( \frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n \). Therefore, for all \( \varepsilon > 0 \), if we take \( N = \frac{1}{\varepsilon} \), then \( n > N \) ensures \( \frac{|n-1|}{n^2-1} < \varepsilon \).

(e) Note that 
\[
\frac{1}{n \cos n} - 0 = \frac{1}{n \cos n} = \frac{1}{n} \cos n \leq \frac{1}{n}
\]
and \( \frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n \). Therefore, for all \( \varepsilon > 0 \), if we take \( N = \frac{1}{\varepsilon} \), then \( n > N \) ensures \( \frac{1}{n \cos n} - 0 < \varepsilon \).

(13) The solution is similar to question (12).

(14) See next page:
(a) State the definition of convergence.

(b) Suppose \( \lim_{n \to \infty} a_n = a \) for \( a \in \mathbb{R} \) and define \( b_n = a_{n+1} \). Using the definition of convergence, prove that \( \lim_{n \to \infty} b_n = a \).

(c) Define a sequence \( s_n \) as follows: \( s_1 = 1 \) and, for \( n \geq 1 \), \( s_{n+1} = \frac{1}{3}(s_n + 1) \). Use induction to prove that \( s_n \geq \frac{1}{2} \) for all \( n \).

(d) Use part (c) to show that the sequence is decreasing.

(e) Prove that \( \lim_{n \to \infty} s_n = s \) for some \( s \in \mathbb{R} \).

(f) Use part (b) and the definition of \( s_n \) to find the value of \( s \).

(a) A sequence of real numbers \((s_n) \subset \mathbb{R}\) is said to converge to a limit \( s \in \mathbb{R} \) if for every \( \epsilon > 0 \), there exists some number \( N \) such that if \( n > N \), then \( |s_n - s| < \epsilon \).

(b) We wish to show that the above definition holds for the sequence \((b_n)\). As always, we begin by letting an arbitrary \( \epsilon > 0 \) be given. Since we know \( \lim_{n \to \infty} a_n = a \), we know that there exists some number \( N \) for which \( n > N \) implies \( |a_n - a| < \epsilon \). Fix any such \( N \) and observe that

\[
|b_n - a| = |a_{n+1} - a| < \epsilon.
\]

(since \( n + 1 > n > N \))

By definition, then, \( \lim_{n \to \infty} b_n = a \).

(c) We wish to show inductively that \( s_n \geq \frac{1}{2} \) for all \( n \).

Base Case: When \( n = 1 \), we have \( s_1 = 1 \geq \frac{1}{2} \).

Inductive Step: We now suppose \( s_n \geq \frac{1}{2} \) and show that \( s_{n+1} \) is as well. We know from the definition of our sequence that

\[
s_{n+1} = \frac{1}{3}(s_n + 1).
\]

Since \( s_n \geq \frac{1}{2} \), we know \( s_n + 1 \geq \frac{3}{2} \) and so,

\[
s_{n+1} = \frac{1}{3}(s_n + 1) \\
\geq \frac{1}{3} \cdot \frac{3}{2} \\
= \frac{1}{2},
\]

completing our inductive step and our proof.

(d) We wish to show \( s_{n+1} \leq s_n \) for all \( n \in \mathbb{N} \). To this end, fix any arbitrary \( n \in \mathbb{N} \). Again, we know

\[
s_{n+1} = \frac{1}{3}(s_n + 1) = \frac{s_n}{3} + \frac{1}{3}.
\]

Since we know \( s_n \geq \frac{1}{2} \), we know

\[
\frac{2}{3} s_n \geq \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.
\]
Therefore,

\[ s_n = \frac{1}{3}s_n + \frac{2}{3}s_n \]
\[ \geq \frac{1}{3}s_n + \frac{1}{3} \]
\[ = s_{n+1}. \]

Since \( n \) was chosen arbitrarily, we may conclude \( s_{n+1} \leq s_n \) for all \( n \).

(c) From part (d), we know that \( s_n \) is a decreasing sequence. From part (c), we know that our decreasing sequence is bounded below. We know that bounded monotone sequences converge, so there must be some \( s \in \mathbb{R} \) so that \( \lim s_n = s \).

(f) From part (b), we may say

\[ s = \lim s_n = \lim s_{n+1} = \lim \frac{1}{3}(s_n + 1). \]

Since \( \lim s_n \) exists and the limits of constants exist, we may appeal to our limit theorems to say

\[ \lim \frac{1}{3}(s_n + 1) = \frac{1}{3} (\lim s_n + 1) = \frac{1}{3} (s + 1). \]

Putting (1) and (2) together, we find

\[ s = \frac{1}{3} (s + 1) \]

or

\[ s = \frac{1}{2}. \]

\( \square \)
(a) State the triangle inequality.

(b) Use the triangle inequality to prove the reverse triangle inequality: for all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$.

(c) Prove that, for any convergent sequence $t_n$, we have

$$\left| \lim_{n \to +\infty} t_n \right| = \lim_{n \to +\infty} |t_n|.$$