Math 117: Homework 2
Due Tuesday, April 20th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1 (Similar to 4.14)

Let $S$ and $T$ be nonempty subsets of $\mathbb{R}$ that are bounded below. Define $S + T = \{s + t : s \in S$ and $t \in T\}$. Prove $\inf S + \inf T = \inf (S + T)$.

Hint: Show that for all $t \in T$, $\inf(S + T) - t$ is a lower bound for $S$. Then show $\inf(S + T) - \inf S$ is a lower bound for $T$.

Question 2* (Similar to 4.8)

Let $A$ and $B$ be nonempty subsets of $\mathbb{R}$ with the following property: $a \leq b$ for all $a \in A$ and $b \in B$.

(a) Show that $A$ is bounded above and $B$ is bounded below.

(b) Prove $\sup A \leq \inf B$.

(c) Given an example of such sets $A$ and $B$ where $A \cap B$ is nonempty.

(d) Give an example of sets $A$ and $B$ where $\sup A = \inf B$ and $A \cap B$ is the empty set. In your explanation, make sure you justify why $\sup A = \inf B$.

Question 3* (Similar to 4.1-4.4)

For each of the sets below, answer the following questions: Is it bounded above? If so, what is its supremum? Is it bounded below? If so, what is its infimum? You do not need to justify your answers.

(a) $[-\sqrt{2}, \sqrt{2}]$

(b) $\{-1, 0, e, \pi\}$

(c) $\{1\}$

(d) $\bigcup_{n=1}^{\infty} [2n - 1, 2n)$

(e) $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$ (Hint: use Question 9)

(f) $\{x \in \mathbb{R} : x^2 < 1\}$

(g) $\bigcap_{n=1}^{\infty} (-1 - \frac{1}{n}, 1 + \frac{1}{n})$ (Hint: use Question 10 to show $\bigcap_{n=1}^{\infty} (-1 - \frac{1}{n}, 1 + \frac{1}{n}) = [-1, 1]$.)
Question 4 (Similar to 4.1-4.4)

Follow the same instructions as in Question 3 for the following sets:

(a) \( \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \) (Hint: use Question 10)

(b) \( \left\{ n + \frac{(-1)^n}{n} : n \in \mathbb{N} \right\} \) (Hint: use Question 9)

(c) \( \{ q \in \mathbb{Q} : q \geq 0 \} \)

(d) \( \{ q \in \mathbb{Q} : q^2 \geq 0 \} \)

(e) \( \cap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) \) (Hint: use Question 10)

(f) \( \{ x \in \mathbb{R} : x^3 \geq 2 \} \)

(g) \( \{ \sin(n\pi) : n \in \mathbb{N} \} \)

Question 5

Given \( a, b \in \mathbb{R} \), consider the set \( (a,b) \). Find the maximum, minimum, supremum, and infimum of the set or state that they do not exist. Justify your answer using the definitions.

Question 6 (Similar to 4.10)

Prove that if \( a < 0 \), then there exists \( n \in \mathbb{N} \) satisfying \( \frac{1}{n} < -a < n \).

Question 7* (Similar to 4.15)

Let \( a,b \in \mathbb{R} \). Show if \( a \leq b + \frac{1}{n} \) for all \( n \in \mathbb{N} \), then \( a \leq b \). (Compare to Question 3 on HW 1.)

Question 8* (Similar to 4.16)

Show that, for all \( a \in \mathbb{R} \), \( \inf \{ q \in \mathbb{Q} : a < q \} = a \).

Question 9* (Similar to 5.4)

The book uses the definition of the real numbers to prove the following corollary. (See Corollary 4.5, p23.)

**COROLLARY 1.** Every nonempty subset \( S \subseteq \mathbb{R} \) that is bounded below has an infimum.

Suppose \( S \) is a nonempty subset of \( \{ x \in \mathbb{R} : x > 0 \} \). Define \( S' = \{ 1/s : s \in S \} \). Prove that if \( S \) is bounded below by some number \( a > 0 \), then \( \sup S' = \frac{1}{\inf S} \). On the other hand, prove that if \( S \) is not bounded below by any \( a > 0 \), then \( \sup S' = +\infty \).

Question 10 (Similar to 5.5)

For any nonempty subset \( A \subseteq \mathbb{R} \), prove that \( \sup A \geq \inf A \). (Do not assume that \( A \) is bounded above or bounded below.)
Question 11* (Similar to 7.4)

Give examples of

(a) A sequence \((x_n)\) of irrational numbers having a limit \(\lim x_n\) that is a rational number.

(b) A sequence \((r_n)\) of rational numbers having a limit \(\lim r_n\) that is an irrational number.

Justify your examples using the definition of a limit.

Question 12* (Similar to 8.1)

Prove the following using the definition of a limit.

(a) \(\lim \left(-\frac{1}{2}\right)^n = 0\)

(b) \(\lim \frac{1}{\sqrt{n}} = 0\)

(c) \(\lim \frac{5n+2}{2n+2} = \frac{5}{2}\)

(d) \(\lim \frac{n-1}{n+1} = 0\)

(e) \(\lim \frac{1}{n} \cos n = 0\)

Question 13 (Similar to 8.2)

For each of the following sequences, find the limit. Justify your answer using the definition of a limit.

(a) \(b_n = \frac{7n-3}{19n+7}\)

(b) \(s_n = \frac{1}{n} \sin(2n)\)

(c) \(a_n = \frac{n}{n^2+5}\)

(d) \(d_n = \frac{3n+4}{5n+3}\)

(e) \(c_n = \frac{14n+3}{17n-5}\)

Question 14*

(a) State the definition of convergence of a sequence.

(b) Suppose \(\lim a_n = a\) for \(a \in \mathbb{R}\) and define \(b_n = a_{n+1}\). Using the definition of convergence, prove that \(\lim b_n = a\).

(c) Define a sequence \(s_n\) as follows: \(s_1 = 1\) and, for \(n \geq 1\), \(s_{n+1} = \frac{1}{3}(s_n + 1)\). Use induction to prove that \(s_n \geq 1/2\) for all \(n\).

(d) Use part (c) to show that the sequence is decreasing.

(e) Prove that \(\lim s_n = s\) for some \(s \in \mathbb{R}\).

(f) Use part (b) and the definition of \(s_n\) to find the value of \(s\).
Question 15

(a) State the reverse triangle inequality.

(b) Use the reverse triangle inequality to prove that for any convergent sequence $t_n$, we have

$$|\lim t_n| = \lim |t_n|.$$