Recall:

Motivation: We know a lot about monotone sequences. What about bounded sequences?

Def (subsequence): Consider a sequence \( s_n \). For any sequence \( n_k \) of natural numbers satisfying \( n_1 < n_2 < n_3 < \ldots \), a sequence of the form \( s_{n_k} \) is a subsequence of \( s_n \).

Def: (subsequential limit) A subsequential limit of a sequence \( s_n \) is any real number or symbol \( +\infty \) or \( -\infty \) that is the limit of some subsequence of \( s_n \).

Thm: If a sequence \( s_n \) converges to a limit, then every subsequence also converges to \( s \).
Thm (main subsequences theorem)
Let $s_n$ be a sequence of real numbers.
(a) Let $t \in \mathbb{R}$
   \[ \text{The set } \{ n : |s_n - t| < \varepsilon \} \text{ is infinite for all } \varepsilon > 0 \]
   if and only if
   \[ t \text{ is a subsequential limit of } s_n. \]
(b) $s_n$ is unbounded above $\iff +\infty$ is a subseq. limit.
(c) $s_n$ is unbounded below $\iff -\infty$ is a subseq. limit.

Why are subsequences important?

Even though not all sequences are monotone

Thm: Every sequence $s_n$ has a monotonic subsequence.

Pf: We will say that the $n^{th}$ element of a sequence is dominant if it is greater than every element that follows, that is $s_n$ is dominant if $s_n > s_m$ for all $m > n$.

Case 1: Suppose $s_n$ has infinitely many dominant elements.
Define $s_{nk}$ to be the subsequence of dominant terms. Then $s_{nk} > s_{nk+1}$ for all $k \in \mathbb{N}$, so $s_{nk}$ is decreasing, hence monotone.

**Case 2:** Suppose $s_n$ has finitely many dominant elements.

- Choose $n_1$ so that $s_{n_1}$ is beyond all of the dominant elements in the sequence.
- Since $s_{n_1}$ is not dominant, there exists $n_2 > n_1$ so that $s_{n_2} \geq s_{n_1}$.
- Since $s_{nk}$ is not dominant, there exists $n_{k+1} > n_k$ so that $s_{nk+1} > s_{nk}$.

Thus we have found a subsequence that is increasing, hence monotone. 

**MAJOR THEOREM 5**

**Thm (Bolzano-Weierstrass):** Every bounded sequence has a convergent subsequence.

**Pf:** If $s_n$ is a bounded sequence, the previous theorem ensures there exists a subsequence $s_{nk}$ that is monotonic (and also bounded). Since all bounded, monotone sequences converge, $s_{nk}$ is convergent.
How do subsequences relate to liminf and limsup?

\[ \liminf_{n \to \infty} S_n \]

Downside: in general \( a_n, b_n \) are not subsequences of \( S_n \).

Upside:

**Thm:** For any sequence \( S_n \), \( \limsup S_n \) and \( \liminf S_n \) are subsequential limits.

*Pf:* First, we will show \( \limsup_{n \to \infty} S_n \) is a subsequential limit.

**CASE 1:** Suppose \( \limsup_{n \to \infty} S_n = -\infty \). Since
\[
\lim_{n \to \infty} S_n \leq \limsup_{n \to \infty} S_n,
\]
then \( \lim_{n \to \infty} S_n = -\infty \), so
\[
\lim_{n \to \infty} S_n = -\infty.
\]

**CASE 2:** Suppose \( \lim_{n \to \infty} S_n = +\infty \), that is \( \lim_{n \to \infty} a_n = +\infty \). Fix arbitrary \( M > 0 \). Then there exists \( N_0 \) s.t. \( N > N_0 \) ensures \( a_n > M \).

Thus \( M \) is not an upper bound of \( \{ S_n : n > N_3 \} \) when \( N > N_0 \), so there exists \( s_{N_1} > M \). Thus \( s_n \) is not bounded above. Hence \( +\infty \) is a subsequential limit.
CASE 3: Suppose \( \limsup_{n \to \infty} s_n = t \) for \( t \in \mathbb{R} \), that is \( \lim_{n \to \infty} a_n = t \). Fix arbitrary \( \varepsilon > 0 \). We will show \( \{ n : t - \varepsilon < s_n < t + \varepsilon \} = \{ n : |s_n - t| < \varepsilon \} \) is infinite.

By defn of convergence of \( a_n \) to \( t \), \( \exists N_0 \) s.t. \( n > N_0 \) ensures \( |a_n - t| < \varepsilon \) \( \Rightarrow \sup \{ s_{n} : n > N_0 \} = a_n < t + \varepsilon \). In particular, for \( N = \lceil N_0 \rceil + 1 \), \( \sup \{ s_{n} : n > \lceil N_0 \rceil \} < t + \varepsilon \). Thus for all \( n > \lceil N_0 \rceil + 1 \), \( s_n < t + \varepsilon \).

Suppose, for the sake of contradiction, that \( \{ n : t - \varepsilon < s_n < t + \varepsilon \} \) is finite. Since we know \( n > \lceil N_0 \rceil \) ensures \( s_n < t + \varepsilon \), there must be \( N_1 > \lceil N_0 \rceil + 1 \) for which \( s_n \geq t - \varepsilon \) for all \( n > N_1 \).

Then \( a_n = \sup \{ s_{n} : n > N_1 \} \leq t - \varepsilon \) for \( n > N_1 \). This implies \( \lim_{n \to \infty} a_n \leq t - \varepsilon \). This contradicts that \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} s_n = t \). Therefore, \( \{ n : t - \varepsilon < s_n < t + \varepsilon \} \) is infinite. Since \( \varepsilon > 0 \) was arbitrary, by main subseq. theorem, \( t \) is a subsequential limit.
Next, we show \( \lim_{n \to \infty} s_n \) is a subsequential limit.

Fact: \( \lim_{n \to \infty} s_n = -\limsup_{n \to \infty} -s_n \)

Thus, by what we've already shown, 
\( \lim_{n \to \infty} s_n \) is a subsequential limit of \(-s_n\)

Fact: \( t \) is a subsequential limit of \( s_n \)
\( \iff \) \(-t \) is a subsequential limit of \(-s_n\)

Thus \( \lim_{n \to \infty} s_n \) is a subsequential limit of \( s_n \) \( \square \)

In fact, \( \limsup_{n \to \infty} s_n \) and \( \liminf_{n \to \infty} s_n \) aren't just any subsequential limit: they are the largest and smallest subsequential limit.

Recall: squeeze lemma
Given \( a_n \leq b_n \leq c_n \) for all \( n \in \mathbb{N} \), if 
\( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n \), then 
\( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n \).

Thm: Let \( S \) denote the set of subsequential limits of \( s_n \), then \( \limsup s_n = \max(S) \) and \( \liminf s_n = \min(S) \).
\textbf{Proof:} By the previous theorem, we have 
\[ \limsup_{n \to \infty} s_n \leq S \quad \text{and} \quad \liminf_{n \to \infty} s_n \geq S, \]
so it suffices to show that, for all \( t \in S \), we have 
\[ \lim_{n \to \infty} s_n \leq t \leq \limsup_{n \to \infty} s_n. \]
Suppose \( \lim_{k \to \infty} s_{n_k} = t \).
Since \( n_k = k \), \( \{s_{n_k} : k > N\} \subseteq \{s_n : n > N\} \)
for any \( N \in \mathbb{R} \). Thus
\[
\begin{align*}
 b_n &= \inf \{s_n : n > N\} \leq \inf \{s_{n_k} : k > N\} \\
&= \sup \{s_{n_k} : k > N\} \leq \sup \{s_n : n > N\} = a_N.
\end{align*}
\]
Sending \( N \to \infty \),
\[
\lim_{n \to \infty} s_n = \lim_{N \to \infty} b_n \leq \lim_{k \to \infty} s_{n_k} = t = \lim_{k \to \infty} s_{n_k} \leq \lim_{N \to \infty} a_N = \limsup_{n \to \infty} s_n. \quad \square
\]