Recall:

Def (bounded function): \( f \) is bounded on \( S \subseteq \text{dom}(f) \) if there exists \( M > 0 \) s.t. \( |f(x)| \leq M \) for all \( x \in S \).

We say \( f \) is bounded if \( f \) is bounded on \( \text{dom}(f) \).

Major Theorem 6:

Thm (cts fns attain max and min): A continuous function \( f \) on a closed interval \( [a, b] \subseteq \text{dom}(f) \) attains its maximum and minimum.

In particular...

(i) its max and min exist (so \( f \) is bounded)

(ii) \( \exists x_{\text{max}}, x_{\text{min}} \in [a, b] \) so that \( f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}}) \) for all \( x \in [a, b] \)

\[ f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}}) \] for all \( x \in [a, b] \)

\( f(x_{\text{min}}) \) is minimum of \( f \) on \( [a, b] \)

\( f(x_{\text{max}}) \) is maximum of \( f \) on \( [a, b] \)

\[ f(a), f(b) \]
Theorem (Intermediate Value Theorem): If $f$ is continuous on an interval $I \subseteq \text{dom}(f)$, then for all $a, b \in I$, if $y$ lies between $f(a)$ and $f(b)$, then there exists $x$ between $a$ and $b$ s.t. $f(x) = y$.

Uniform continuity

Recall:

Theorem ($\varepsilon$-$\delta$ characterization of continuity): Given $f$ and $x_0 \in \text{dom}(f)$, $f$ is cts at $x_0$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

In general, the choice of $\delta$ depends on both $\varepsilon$ and $x_0$.

Example:

$f(x) = \frac{1}{x}$

Diagram showing how $\varepsilon$-$\delta$ values are chosen for different points on the graph of $f(x)$.
However, there are some functions for which $S$ only depends on $\varepsilon$ and not $x_0$.

**Definition (uniformly cts):** Given a function $f$ and $S \subseteq \text{dom}\, f$, $f$ is uniformly cts on $S$ if

for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

**Remark:**
- $f$ is uniformly continuous on $S \implies f$ is cts on $S$.
- $f$ is cts on $S \implies f$ is uniformly continuous on $S$.

**Example:** Consider $f(x) = \frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$.

- $f$ is continuous on $\mathbb{R} \setminus \{0\}$.

- $f$ is not uniformly continuous on $\mathbb{R} \setminus \{0\}$.

**Proof:** Assume, for the sake of contradiction, that $f$ is uniformly continuous on $\mathbb{R} \setminus \{0\}$. Let $\varepsilon = 1$. Then there exists $\delta > 0$ so that $x, y \in S$ and $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon = 1$.

Choose $n > \frac{1}{\delta}$ and let $x = \frac{1}{n}$, $y = \frac{1}{n+1}$. Then,
\[ |x-y| = \left| \frac{1}{n} - \frac{1}{n+1} \right| = \frac{1}{n(n+1)} < \delta \quad \text{and} \quad |f(x) - f(y)| = |n-(n+1)| = 1 \geq 3. \]

This is a contradiction. \( \square \)

**Scratchwork:** \( x = \frac{1}{n}, \ y = \frac{1}{n+1}, \ |f(x) - f(y)| = |n-(n+1)| = 1 \)

\[ |x-y| < \delta \iff \left| \frac{1}{n} - \frac{1}{n+1} \right| < \delta \iff \left| \frac{1}{n(n+1)} - \frac{n}{n(n+1)} \right| < \delta \iff \frac{1}{n} < \delta \iff \frac{1}{\delta} < n \]

\( f \) is uniformly continuous on \( [a,b] \subseteq \mathbb{R} \setminus \{0\} \), for any \( a \leq b \).

**Proof:** Fix \( \varepsilon > 0 \). Let \( \delta = 3 \cdot \min \{a^2, b^2\} \). If \( \frac{x}{y} \in [a,b] \), then \( |x-y| = |x| |y| = \min \{a^2, b^2\} \).

\[ |f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x-y|}{xy} < \frac{\delta}{\min \{a^2, b^2\}} = 3. \] \( \square \)

**Scratchwork:**

\[ |f(x) - f(y)| < 3 \iff \left| \frac{1}{x} - \frac{1}{y} \right| < 3 \iff \frac{|x-y|}{xy} < 3 \iff \frac{|x-y|}{\min \{a^2, b^2\}} < 3 \iff 3 \cdot \min \{a^2, b^2\} < 3 \iff 3 \cdot \min \{a^2, b^2\} < 3 \]
In the previous example we showed that \( f(x) = \frac{1}{x} \)

- is continuous
- is not uniformly continuous on \( \text{dom}(f) \)
- is uniformly continuous on \([a, b] \subset \text{dom}(f)\)

In fact, this type of result is true for any cts \( fn \) on any closed interval.

**Thm (on closed interval, cts \(\Rightarrow\) unif cts):** If \( f \) is cts on a closed interval \([a, b] \subset \text{dom}(f)\), then \( f \) is uniformly cts on \([a, b]\).

Recall:

- If \( s_n \in [a, b] \) \( \forall n \) and \( s_n \) converged, then \( \lim s_n \in [a, b] \)
- If \( s_n \in (a, b) \) \( \forall n \) and \( s_n \) converged, we don't know that \( \lim s_n \in (a, b) \). (For example, \( s_n = b - \frac{1}{n} \).)

**Pf:** Assume, for the sake of contradiction, that \( f \) is not uniformly continuous on \([a, b]\), that is there exists \( \varepsilon > 0 \) so that for all \( \delta > 0 \), there exist \( x, y \in [a, b] \) with \( |x - y| < \delta \) and \( |f(x) - f(y)| \geq \varepsilon \).
In particular, for any \( n \in \mathbb{N} \), there exist \( x_n, y_n \in [a, b] \) with \( |x_n - y_n| < \frac{1}{n} \) and 
\[ |f(x_n) - f(y_n)| \leq \varepsilon. \]

Since \( x_n, y_n \in [a, b] \), for \( M = \max\{|a|, |b|, |x_n|, |y_n|\} \) \( |x_n| < M \) and \( |y_n| < M \) for all \( n \in \mathbb{N} \), so they are bounded sequences. Thus, by Bolzano–Weierstrass, there exist convergent subsequences \( x_{n_k} \) and \( y_{n_k} \) with limits \( x_0 = \lim_{n \to \infty} x_{n_k} \) and \( y_0 = \lim_{n \to \infty} y_{n_k} \) satisfying \( x_0, y_0 \in [a, b] \).

Fix \( \varepsilon > 0 \) and choose \( K \) s.t. \( k \geq K \), 
\[ |x_0 - x_{n_k}| < \frac{\varepsilon}{3} \text{ and } |y_0 - y_{n_k}| < \frac{\varepsilon}{3}. \]
Then by the triangle inequality, for \( k \geq K \)
\[ |x_0 - y_0| = |x_0 - x_{n_k} + x_{n_k} - y_{n_k} + y_{n_k} - y_0| \]
\[ \leq \frac{\varepsilon}{3} + \frac{1}{n_k} + \frac{\varepsilon}{3} \]
\[ \leq 2\varepsilon + \frac{1}{k} \]
Since \( \varepsilon > 0 \) and \( k \geq K \) were arbitrary, \( x_0 = y_0 \).

Since \( f \) is continuous, \( \lim_{k \to \infty} f(x_{n_k}) = f(x_0) \)
and \( \lim_{k \to \infty} f(y_{n_k}) = f(y_0) = f(x_0) \). Thus
\[ \lim_{k \to \infty} f(x_{n_k}) - f(y_{n_k}) = f(x_0) - f(x_0) = 0. \]This
contradicts that \( |f(x_n) - f(y_n)| \leq \varepsilon \) for all \( n \in \mathbb{N} \).
Thus $f$ is uniformly continuous on $[a, b]$. \( D \)

**Remark:** It is not true that if $f$ is a continuous function then it is uniformly continuous on any open interval $(a, b) \subseteq \text{dom}(f)$.

To see this, consider $f(x) = \frac{1}{x}$ and $(0, 1) \subseteq \text{dom}(f)$. 
Now: key property of uniformly cts fn's...

Think back to continuous functions...

"Continuous functions send convergent sequences to convergent sequences."

That is, if \( \lim_{n \to \infty} x_n = x_0 \), then \( \lim_{n \to \infty} f(x_n) = f(x_0) \),

\[ \text{dom}(f) \]

Wait a second... if we consider the continuous function \( f(x) = \frac{1}{x} \) and the convergent sequence \( x_n = \frac{1}{n} \), \( f \) doesn't "send convergent sequence to convergent sequence" since \( f(x_n) = n \) doesn't converge.

\[ \lim_{n \to \infty} x_n = 0 \notin \text{dom}(f). \]

Remarkably, uniformly continuous functions satisfy (1) without the additional assumptions in red.
**Theorem (uniform continuity and convergence):** If $f$ is uniformly continuous function on a set $S$ and $\{s_n\}$ is a convergent sequence, then $f(\{s_n\})$ is a convergent sequence.

**Proof:** Fix $\varepsilon > 0$. Since $f$ is uniformly continuous on $S$, there exists $\delta > 0$ so that if $x, y \in S$ and $|x - y| < \delta$, imply $|f(x) - f(y)| < \varepsilon$.

We will show $f(\{s_n\})$ is a Cauchy sequence. Since $\{s_n\}$ is a convergent sequence, it is Cauchy, so there exists $N$ s.t. $n, m \geq N$, $|s_n - s_m| < \delta$. By choice of $\delta$, $|f(s_n) - f(s_m)| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, $f(\{s_n\})$ is Cauchy, hence convergent. \(\square\)