Lecture 1
© Katy Craig, 2021

Math 117
Prof. Katy Craig

Course goal: transition to higher level math

- You will have to back up your claims using clear, logical arguments
- You must be able to precisely state important definitions and theorems.
- If you come across something in class or in the book that doesn’t make sense...
  1. Learn all relevant definitions
  2. Ask me or TA for help
  3. Be patient. If you stay on top of learning definitions, things will start to make sense. Otherwise, things will become more and more confusing.
Why analysis? What is analysis?

→ take everything you learned in Calculus and put it on rigorous mathematical footing.

→ analysis is the mathematics of approximation, a key link between mathematics in the real world and mathematics in the fake world.

Real world observations

Numerical simulations

Mathematical Model

Predictions
Numbers?
- Natural numbers \( \mathbb{N} = \{1, 2, 3, 4, \ldots\} \)
- Integers \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\} \)
- Rational numbers \( \mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\} \)
- Real numbers \( \mathbb{R} = \text{what to put here?} \)

Intuitively, \( \mathbb{R} \) is all the numbers on a number line.

Developing a precise definition of \( \mathbb{R} \) will be our first main result.

---

**Key properties of these classes of numbers**

**Property 1: Inductive Characterization of \( \mathbb{N} \)**
If a subset \( S \subseteq \mathbb{N} \) satisfies
(i) \( 1 \in S \)
(ii) if \( n \in S \), then \( n+1 \in S \)
then \( S = \mathbb{N} \).

Ex: \( S = \{2, 3, 4, \ldots\} \) fails (i)
\( S = \{1, 2, 3, 5, \ldots\} \) fails (ii)
IMPORTANT REMARK

This is the basis of proof by induction.

Suppose \( \{P_1, P_2, P_3, \ldots\} = \{P_k : k \in \mathbb{N}\} \) is a list of statements.

\[ P_k = \{k+25 \text{ is an integer}\} \]
\[ P_k = \{\text{Katy would like } k \text{ cookies}\} \]

Suppose you can prove that

(a) \( P_1 \) is true \( \Leftarrow \) Base case

(b) if \( P_n \) is true, then \( P_{n+1} \) is true \( \Leftarrow \) Inductive Step

What does this tell us about \( S = \{k \in \mathbb{N} : P_k \text{ is true}\} \)?

(a) ensures \( 1 \in S \)
(b) if \( n \in S \), then \( n+1 \in S \)

Property 1 ensures that \( S = \mathbb{N} \).

In other words, \( P_k \) is true for all \( k \in \mathbb{N} \).

The previous remark shows that the fact that proof by induction works is a consequence of Property 1.
Exercise 1

Prove by induction that, for all \( n \in \mathbb{N} \),

\[
1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 2 - \frac{1}{2^n}.
\]
**Property 2: Ordered Sets**

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are **ordered sets**, that is

for any elements $a, b,$ and $c$...

(i) $a \leq a$  
(ii) If $a \leq b$ and $b \leq a$, then $a = b$  
(iii) If $a \leq b$ and $b \leq c$, then $a \leq c$  
(iv) Either $a \leq b$ or $b \leq a$

"$\leq$" also satisfies the following properties wrt "$+$" and "$\cdot$":

(v) If $a \leq b$, then $a + c \leq b + c$
(vi) If $a \leq b$ and $c \geq 0$, then $a \cdot c \leq b \cdot c$

If $a \leq b$ and $a \neq b$, we write $a < b$.

<table>
<thead>
<tr>
<th>Exercise 2</th>
</tr>
</thead>
</table>

Prove the following statements using properties (i)-(vi) above. Suppose $a, b \in \mathbb{Q}$.

1. If $a \leq b$, then $-b \leq -a$.
2. $0 \leq a^2$ for all $a$. 
Property 3: \( \mathbb{Q} \) is dense in \( \mathbb{Q} \)

Prop: For any \( p, q \in \mathbb{Q} \) with \( p < q \), there exists \( r \in \mathbb{Q} \) satisfying \( p < r < q \).

"Between any two rational numbers, there is a rational number."

Proof:

Let \( r = \frac{p + q}{2} \).

- Since \( p, q \in \mathbb{Q} \), \( \exists \) \( m_p, n_p, m_q, n_q \in \mathbb{Z} \) with \( n_p, n_q \neq 0 \) so that \( p = \frac{m_p}{n_p} \) and \( q = \frac{m_q}{n_q} \).

  Consequently,
  \[
  r = \frac{p + q}{2} = \frac{1}{2} \left( \frac{m_p}{n_p} + \frac{m_q}{n_q} \right) = \frac{m_p n_q + m_q n_p}{2 n_p n_q},
  \]

  where \( \frac{m_p n_q + m_q n_p}{2 n_p n_q} \) is an integer.

  Thus, \( r \in \mathbb{Q} \).

- Note that \( p < q \implies \frac{p}{2} < \frac{q}{2} \).

  Thus, \( p = \frac{p}{2} + \frac{p}{2} \leq \frac{p}{2} + \frac{q}{2} \leq \frac{q}{2} + \frac{q}{2} = q \).

  \( \therefore \)
Property 4: Absolute Value and Distance

We may take the absolute value of any number in \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \).

**Def:** \( |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases} \)

**Thm (basic properties of \( |\cdot| \)):** For all \( a, b \in \mathbb{R} \),

(i) \( |a| \geq 0 \)

(ii) \( |ab| = |a| \cdot |b| \)

(iii) \( |a+b| \leq |a| + |b| \) \( \left\langle \text{triangle inequality} \right\rangle \)

**Exercise 3**

Prove this theorem using Exercise 2 and properties (i)-(vi) of ordered sets.

We can use the absolute value to define a notion of distance between any two elements of \( \mathbb{R} \).

**Def:** \( \text{dist}(a, b) = |a - b| \).
Property 5: Strict Containment

\[ \mathbb{N} \not\subseteq \mathbb{Z} \subseteq \mathbb{Q} \not\subseteq \mathbb{R} \]

"obvious" \hspace{1cm} "not obvious: we haven't defined \( \mathbb{R} \)!

Recall: Given two sets \( A, B \),
- \( A \subseteq B \) if, \( \forall a \in A, a \in B \).
- \( A \not\subseteq B \) if \( A \not\subseteq B \) and \( \exists b \in B \) s.t. \( b \not\in A \).
  \[ \iff A \not\subseteq B \text{ and } A \neq B \]

\underline{Spoiler}: it will be part of our def'n of \( \mathbb{R} \) that \( \mathbb{Q} \subseteq \mathbb{R} \).

Now, we will show \( \exists b \in \mathbb{R} \) s.t. \( b \not\in \mathbb{Q} \).

\underline{Prop}: \( \sqrt{2} \not\in \mathbb{Q} \)

To prove this, we will first prove a lemma

\underline{Lemma}: Let \( x \in \mathbb{Z} \). If \( x^2 \) is even, then \( x \) is even.
Assume for the sake of contradiction that $x$ is odd.

Then $x - 1$ is even, so $\exists y \in \mathbb{Z}$ s.t. $x - 1 = 2y$.

Thus $x^2 = (1 + 2y)^2 = 1 + 4y + 4y^2$, so $x^2$ is odd.

This is a contradiction.

Thus $x$ is even. \(\square\)

Assume for the sake of contradiction that $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = \frac{m}{n}$ for $m, n \in \mathbb{Z}$, $n \neq 0$.

We may choose $m$ and $n$ so they aren't both even.

Squaring, we obtain $2 = \frac{m^2}{n^2} \Rightarrow 2n^2 = m^2$.

By Lemma, $m^2$ even implies $m$ is even, that is $\exists y \in \mathbb{Z}$ s.t. $m = 2y$. 
Substituting into $\circ$, $2n^2 = (2y)^2 = 4y^2 \Rightarrow n^2 = 2y^2$.

By Lemma, $n^2$ even implies $n$ is even.

This is a contradiction.

Thus, $\sqrt{2} \notin \mathbb{Q}$. $\blacksquare$