Recall:

**MAJOR RESULT #1**

**Thm (Archimedean Property):** If \( a, b \in \mathbb{R} \) satisfy \( a > 0 \) and \( b > 0 \), then there exists \( n \in \mathbb{N} \) so that \( na > b \).

**MAJOR RESULT #2**

**Thm (\( \mathbb{Q} \) is dense in \( \mathbb{R} \)):** If \( a, b \in \mathbb{R} \) with \( a < b \), there exists \( r \in \mathbb{Q} \) s.t. \( a < r < b \).

We also defined \( \sup(S) \) and \( \inf(S) \) even when \( S \) is unbounded above or below.

Given \( S \subseteq \mathbb{R} \) nonempty,

\[
\sup(S) = \begin{cases} 
\text{least upper bound of } S \\
+\infty 
\end{cases} \quad \text{if } S \text{ is bounded above}
\]

\[
\inf(S) = \begin{cases} 
\text{greatest lower bound of } S \\
-\infty 
\end{cases} \quad \text{if } S \text{ is bounded below}
\]
Ch 2: Sequences

Recall: functions

**Def (sequence):** A sequence is a function whose domain is a set of the form \( \{ m, m+1, m+2, \ldots \} \) for some \( m \in \mathbb{Z} \). We will study sequences whose range is \( \mathbb{R} \).

Typically, the domain of a sequence will be either \( \{0, 1, 2, 3, \ldots \} \) or \( \{1, 2, 3, \ldots \} \).

**Remark:**

To emphasize that a sequence is a special type of function...

instead of writing \( f(n) \), we write \( s_n \)

We'll often specify a sequence by listing its values in order, \( \{ s_1, s_2, s_3, \ldots \} \).
Ex: If $s_n = \frac{1}{n}$ for $n \geq 1$, the sequence is $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

- If $s_n = (-1)^n$ for $n \geq 0$, the sequence is $(1, -1, 1, -1, \ldots)$

Heuristically, a sequence "converges" to some limit $s \in \mathbb{R}$ if the values of $s_n$ stay close to $s$ for large $n$.

Ex: We expect $s_n = \frac{1}{n}$ converges to 0.

We expect $s_n = (-1)^n$ doesn't converge.
Def (convergence):
- A sequence $s_n$ of real numbers converges to some $s \in \mathbb{R}$ provided that
  
  \[
  \text{for all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ so that } n > N \text{ ensures } |s_n - s| < \varepsilon.
  \]

- The number $s$ is the limit of $s_n$, and we write $\lim_{n \to \infty} s_n = s$ or $s_n \to s$.

- A sequence that does not converge to any $s \in \mathbb{R}$ it is said to diverge.

Remark:
- Recall: $|b| < a \iff -a < b < a$
- Thus $|s_n - s| < \varepsilon \iff -\varepsilon < s_n - s < \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon$
- $N$ can depend on $\varepsilon$.
Ex: Consider the sequence \( s_n = \frac{1}{n^2} \). We expect that \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \). Let's prove this!

**Scratchwork:** \( |\frac{1}{n^2} - 0| < \epsilon \iff \frac{1}{n^2} < \epsilon \iff \frac{1}{\sqrt{\epsilon}} < n \)

**Proof:** Fix arbitrary \( \epsilon > 0 \). Let \( N = \frac{1}{\sqrt{\epsilon}} \). Then for \( n > N \), we have

\[
\frac{1}{n^2} \leq \frac{1}{\epsilon} \iff \frac{1}{n^2} < \epsilon \iff |\frac{1}{n^2} - 0| < \epsilon.
\]
Thus \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \). \( \square \)

**Remark:** We could have picked \( N \) to be any number \( \geq \frac{1}{\sqrt{\epsilon}} \), e.g. \( N = \frac{2}{\epsilon^2} \), \( N = \frac{1}{\sqrt{\epsilon}} + 1 \), etc.

Ex: Consider the sequence \( s_n = (-1)^n \). We expect that this sequence does not converge. Let's prove it.

**Proof:** Assume, for the sake of contradiction, that \( (-1)^n \) converges to \( s \in \mathbb{R} \). By defn of convergence, for all \( \epsilon > 0 \), there exists \( N \) so that \( n > N \), \( |(-1)^n - s| < \epsilon \).
Let $\varepsilon = 1$ and choose $N$ so that $n > N$ ensures $|(-1)^n - s| < 1 \iff s - 1 < (-1)^n < s + 1$.

For $n$ even, this implies $1 < s + 1 \implies 0 < s$. For $n$ odd, this implies $s - 1 < -1 \implies s < 0$. This is a contradiction. Thus, $(-1)^n$ diverges. □

Ex: Consider the sequence $s_n = \frac{2n - 1}{3n + 2}$.
What is the limit?

| Scratchwork: |

$s_n = \frac{2n - 1}{3n + 2} = \frac{2 - \frac{1}{n}}{3 + \frac{2}{n}}$

$|s_n - \frac{2}{3}| < 3 \iff |\frac{2n - 1}{3n + 2} - \frac{2}{3}| < 3 \iff |\frac{6n^2 - 6n - 7}{3(3n + 2)}| < 3$

$\iff |\frac{-7}{3(3n + 2)}| < 3 \iff \frac{7}{3(3n + 2)} < 3$

$\iff \frac{7}{3n} < 3 \iff \frac{1}{n} < 3 \iff \frac{1}{3} < n$
**Proof:**

Fix $\varepsilon > 0$ arbitrary and let $N = \frac{1}{\varepsilon}$. Then, if $n > N$, we have

$$\frac{1}{\varepsilon} < n \Rightarrow \frac{1}{3(3n+2)} < \varepsilon \Leftrightarrow \left| \frac{6n-3-6n-4}{3(3n+2)} \right| < \varepsilon \Leftrightarrow \left| S_n - \frac{2}{3} \right| < \varepsilon.$$

Therefore, $\lim_{n \to \infty} S_n = \frac{2}{3}$. \qed
A special type of sequence is a...

**Def (bounded sequence):** A sequence $s_n$ is bounded if there exists $M \in \mathbb{R}$ s.t. $|s_n| \leq M$ for all $n$.

**Remark:** A sequence is bounded iff the set $S = \{s_n : n \in \mathbb{N}\}$ is bounded.

**Thm:** Convergent sequences are bounded.

**Example:**
- $s_n = (-1)^n$ gives $S = \{-1, 1, -1, 1, \ldots\}$
- $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ gives $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

Idea of proof:

![Graph of sequence $s_n$ with $\varepsilon$ and $N$](attachment:image.png)
Proof:
Suppose $s_n$ is a convergent sequence with limits. By the definition of convergence, for all $\varepsilon > 0$, there exists $N$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.

Let $\varepsilon = \frac{1}{\sqrt{2}}$. Then there exists $N$ so that $n > N$, $|s_n - s| < \frac{1}{\sqrt{2}}$.

Since $|s_n| - |s| \leq |s_n - s| < \frac{1}{\sqrt{2}}$, so $|s_n| < |s| + \frac{1}{\sqrt{2}}$ for all $n > N$. $\sqrt{2} = \max \{ n : n \in \mathbb{N}, s_n < s + \frac{1}{\sqrt{2}} \}$

Define $M = \max \{ |s_1|, |s_2|, ..., |s_n|, |s| + \frac{1}{\sqrt{2}} \}$. Then $|s_n| \leq M$ for all $n$, so $s_n$ is a bounded sequence. $\square$