Recall:

**Def**: (Cauchy sequence) A sequence $s_n$ is a *Cauchy sequence* if, for all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ s.t. $m, n > N$ ensures $|s_n - s_m| < \varepsilon$.

"Beyond this threshold, elements of the sequence get close to each other."

**Major Theorem #4**

**Thm**: A sequence is convergent iff it is Cauchy.

**Types of Sequences**
Goal: we know a lot about monotone sequences... what can we say about bounded sequences.

First, recall...

**Def (sequence):** A sequence is a function whose domain is a set of the form \( \{m, m+1, m+2, \ldots\} \) for some \( m \in \mathbb{Z} \). We study sequences whose range is \( \mathbb{R} \).

**Remark:** While we could write \( s(n) \), we use \( S_n \) to emphasize that sequences are a special type of functions.

Now, we will define the notion of subsequence.

**Def (subsequence):** Consider a sequence \( s_n \). For any sequence \( n_k \) of natural numbers satisfying \( n_1 < n_2 < n_3 < \ldots \), a sequence of the form \( S_{n_k} \) is a subsequence of \( s_n \).

**Remark:** We could write \( s_n \) as \( s(n) \), \( n_k \) as \( n(k) \), and \( S_{n_k} \) as \( s(n(k)) \).
Informally, a subsequence is any infinite collection of elements from the original sequence, listed in order.

Ex 1: $s_n = (-1, 2, -3, 4, \ldots, (-1)^n n, \ldots)$

$k=1$ $k=2$

$s_{n_k} = (-1, -3, -5, \ldots, (-1)^{2k-1} (2k-1), \ldots)$

$k=1$ $k=2$ $k=3$

$n_k = (1, 3, 5, \ldots, 2k-1, \ldots)$

Note that

$a_n = \sup \{ s_n : n > N \} = (+\infty, +\infty, \ldots, +\infty, \ldots)$

$b_n = \inf \{ s_n : n > N \} = (-\infty, -\infty, \ldots, -\infty, \ldots)$

Ex 2: $s_n = (1, \frac{1}{2}, 3, \frac{1}{4}, \ldots, n, (-1)^{n+1} n, \ldots)$

$k=1$ $k=2$ $k=3$

$s_{n_k} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, (2k)^{-1}, \ldots)$

$k=1$ $k=2$ $k=3$

$n_k = (2, 4, 6, \ldots, 2k, \ldots)$

$a_n = \sup \{ s_n : n > N \} = (+\infty, +\infty, \ldots)$

$b_n = \inf \{ s_n : n > N \} = (0, 0, \ldots)$
Limits of Subsequences

Lemma: Given a sequence \( s_n, n \in \mathbb{N} \), if \( s_{n_k} \) is a subsequence, then \( n_k \geq k \) for all \( k \in \mathbb{N} \).

Pf: Base case: When \( k = 1 \), \( n_1 \geq 1 \) since \( n_k \in \mathbb{N} \) for all \( k \).

Inductive step: Assume \( n_{k-1} \geq k-1 \). Since \( n_k > n_{k-1} \), we have \( n_k > n_{k-1} + 1 \geq k \). \( \square \)

Def: (Subsequential limit) A sub sequential limit of a sequence \( s_n \) is any real number or symbol \( +\infty \) or \( -\infty \) that is the limit of some subsequence of \( s_n \).

Ex: \( s_n = (1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots) \)

0 and \( +\infty \) are subsequential limits

Thm: If a sequence \( s_n \) converges to a limit \( s \), then every subsequence also converges to \( s \).
Let $s_{n_k}$ be an arbitrary subsequence of $s_n$. Fix $\varepsilon > 0$. Since $\lim_{n \to \infty} s_n = s$, $\exists \, N \text{ s.t. } n > N$ ensures $|s_n - s| < \varepsilon$. If $k > N$, then $n_k = k > N$, so $|s_{n_k} - s| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $\lim_{k \to \infty} s_{n_k} = s$.

Ex: $s_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

$\{0^2\}$ is the set of all subsequential limits.

**Theorem (main subsequences theorem):** Let $s_n$ be a sequence of real numbers.

(a) Let $t \in \mathbb{R}$

[The set $\{n : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$]

if and only if

[t is a subsequential limit of $s_n$.]

(b) $s_n$ is unbounded above $\iff +\infty$ is a subseq. limit.

(c) $s_n$ is unbounded below $\iff -\infty$ is a subseq. limit.

**Mental image (a):**

\[ s_n \]
\[ \vdots \]
\[ t \]
\[ \varepsilon \]
\[ n \]
Mental image (b) + (c)

Lemma: If $s_n$ is unbounded above, the set $\{n: s_n > M^3\}$ is infinite for all $M > 0$.

Pf: Assume, for the sake of contradiction, that there exists $M > 0$ for which $\{n: s_n > M^3\}$ is finite. Define

$$s_{\text{max}} = \max \{n: s_n > M^3\}.$$

Then define $\tilde{M} = \max \{s_{\text{max}}, M^3\}$

- if $s_n > \tilde{M}$, $s_n = s_{\text{max}} \leq \tilde{M}$
- if $s_n \leq \tilde{M}$, $s_n \leq \tilde{M}$

Thus, for all $n \in \mathbb{N}$, $s_n \leq \tilde{M}$, so $s_n$ is bounded above, which is a contradiction. $\square$
Proof of Main Subsequences Theorem

(a) Suppose \[ \text{The set } \{n \mid |s_n-t| < \varepsilon \} \text{ is infinite for all } \varepsilon > 0 \].

We can construct a subsequence of \( s_n \) in the following way:

Choose \( s_{n_1} \) so that \(|s_{n_1}-t| < 1\).
Choose \( s_{n_2} \) so that \(|s_{n_2}-t| < \frac{1}{2} \) and \( n_2 > n_1 \).
Choose \( s_{n_k} \) so that \(|s_{n_k}-t| < \frac{1}{k} \) and \( n_k > n_{k-1} \).

Note that \(|s_{n_k}-t| < \frac{1}{k} \iff t - \frac{1}{k} < s_{n_k} < t + \frac{1}{k} \)
for all \( k \in \mathbb{N} \). So by the squeeze lemma,
\( t = \lim_{k \to \infty} s_{n_k} \leq t \), so \( \lim_{k \to \infty} s_{n_k} = t \) and \( t \) is a subsequential limit.

Now, suppose \( t \) is a subsequential limit of \( s_n \).

Fix \( \varepsilon > 0 \). Since there exists a subsequence \( s_{n_k} \) that converges to \( t \), there exists \( N \) s.t. \( k > N \) ensures \(|s_{n_k}-t| < \varepsilon \).
Therefore, \( \{n_k \mid k > N \} \subseteq \{n \mid |s_n-t| < \varepsilon \} \).
Since \( \{n_k \mid k > N \} \) is infinite, so is \( \{n \mid |s_n-t| < \varepsilon \} \).
Suppose \( [s_n \text{ is unbounded above}] \).
By the lemma, for all \( M > 0 \), \( \exists n : s_n > M^2 \) is infinite. Hence, we may construct a subsequence as follows.
Choose \( n_1 \) so that \( s_{n_1} > 1 \).
Choose \( n_2 \) so that \( s_{n_2} > 2 \) and \( n_2 > n_1 \).
Choose \( n_k \) so that \( s_{n_k} > k \) and \( n_k > n_{k-1} \).

Fix \( \tilde{M} > 0 \). For \( k > \tilde{M} \), \( s_{n_k} > k > \tilde{M} \).
Since \( \tilde{M} \) was arbitrary, \( \lim_{k \to \infty} s_{n_k} = +\infty \).
Thue, \( +\infty \) is a subsequential limit.

Suppose \( [+\infty \text{ is a subsequential limit}] \).
Assume, for the sake of contradiction, that \( s_n \) is bounded above, that is, there exists \( M > 0 \) s.t. \( s_n \leq M \) for all \( n \in \mathbb{N} \). Take \( s_{n_k} \) s.t. \( \lim_{k \to \infty} s_{n_k} = +\infty \).
Then \( s_{n_k} \leq M \) for all \( k \in \mathbb{N} \). This is a contradiction.

(c) Note that
\( [s_n \text{ is unbounded below}] \)
\( \Box \)
[-\infty is unbounded above]
\[ \exists (b) \]
[+\infty is a subsequential limit of -\infty]
\[ \exists \]
[-\infty is a subsequential limit of \infty]