Math 117: Practice Final Exam

Question 1

This question will guide you through the proof of the comparison test for series of real numbers:

THEOREM 1 (Comparison Test). Let \( \sum_{k=1}^{\infty} a_k \) be a series of real numbers with \( a_k \geq 0 \) for all \( k \in \mathbb{N} \).

(i) If \( \sum_{k=1}^{\infty} a_k \) converges and \( |b_k| \leq a_k \) for all \( k \in \mathbb{N} \), then \( \sum_{k=1}^{\infty} b_k \) converges.

(ii) If \( \sum_{k=1}^{\infty} a_k = +\infty \) and \( b_k \geq a_k \) for all \( k \in \mathbb{N} \), then \( \sum_{k=1}^{\infty} b_k = +\infty \).

Recall that we proved the second part of this theorem in Lecture 11. Consequently, this problem will take you through the steps to prove the first part.

(a) Prove that, for all \( n \leq m \),

\[
\left| \sum_{k=n}^{m} b_k \right| \leq \sum_{k=n}^{m} a_k.
\]

(b) Use part (a) to prove that \( \sum_{k=1}^{\infty} b_k \) satisfies the Cauchy criterion, from Question 7 on Practice Quiz 3. (Hint: do you know if \( \sum_{k=1}^{\infty} a_k \) satisfies the Cauchy criterion?)

(c) Use part (b) to prove part (i) of the theorem.

Question 2

This question will guide you through the proof of the root test for series of real numbers.

THEOREM 2 (Root Test). Consider a series \( \sum_{k=1}^{\infty} a_k \) and define \( \alpha = \limsup_{k \to +\infty} |a_k|^{1/k} \).

(i) If \( \alpha < 1 \), \( \sum_{k=1}^{\infty} a_k \) converges.

(ii) If \( \alpha > 1 \), \( \sum_{k=1}^{\infty} a_k \) diverges.

(1) First we consider the case \( \alpha < 1 \).

(a) Prove that there exists \( \epsilon > 0 \) so that \( \alpha + \epsilon < 1 \). (Hint: the proof is very short.)

(b) Show that there exists \( N \) so that

\[
\alpha - \epsilon < \sup\{ |a_k|^{1/k} : k > N \} < \alpha - \epsilon
\]

(c) Use part (1a) and the comparison test to the geometric series \( \sum_{k=N+1}^{\infty} (\alpha + \epsilon)^k \) to conclude that the series \( \sum_{k=N+1}^{\infty} |a_k| \) converges.

(d) Conclude that the original series \( \sum_{k=1}^{\infty} a_k \) converges. (Hint: use the result of Q2.2 from Quiz 3.)

(2) Next, we consider the case \( \alpha > 1 \).

(a) Prove that there exists a subsequence so that \( \lim_{i \to +\infty} |a_{k_i}|^{1/k_i} = \alpha \).

(b) Use part (2a) to show that \( |a_k| > 1 \) for infinitely many \( k \in \mathbb{N} \).

(c) Use part (2b) to show that \( \lim_{k \to +\infty} a_k \neq 0 \).

(d) Use Corollary 2 in Practice Quiz 3, Question 7, to conclude that \( \sum_{k=1}^{\infty} a_k \) diverges.
**Question 3**

This question will guide you through the proof of the following theorem:

**THEOREM 3** (Radius of Convergence of Power Series). Given a power series \( \sum_{k=1}^{\infty} a_k x^k \), let \( \beta = \limsup_{k \to +\infty} |a_k|^{1/k} \) and define

\[
R = \begin{cases} 
+\infty & \text{if } \beta = 0, \\
\frac{1}{\beta} & \text{if } \beta \in (0, +\infty), \\
0 & \text{if } \beta = +\infty.
\end{cases}
\]

Then the power series converges pointwise on \((-R, R)\) and diverges for any \(x \in (-\infty, R) \cup (R, +\infty)\).

1. Define the function \( \alpha(x) = \limsup_{k \to +\infty} |a_k x^k|^{1/k} \). Prove that \( \alpha(x) = \beta x \) for all \(x \in \mathbb{R}\).

2. Apply the Root Test from Question 2 to prove the theorem.

**Question 4**

Let \( f_n \) be a sequence of continuous functions on \([a, b]\) that converges uniformly to \( f \) on \([a, b]\). Suppose \( x_n \in [a, b] \) and \( \lim_{n \to +\infty} x_n = x \). Prove that \( \lim_{n \to +\infty} f_n(x_n) = f(x) \).

**Question 5**

Let \( f_n(x) = \frac{x^n}{n} \). Prove that \( f_n \) is uniformly convergent on \([-1, 1]\).

**Question 6**

(a) Prove that if \( \sum_{k=1}^{\infty} |a_k| < +\infty \), then \( \sum_{k=1}^{\infty} a_k x^k \) converges uniformly on \([-1, 1]\) to a continuous function.

(b) Does \( \sum_{k=1}^{\infty} \frac{x^k}{k^2} \) converge to a continuous function on \([-1, 1]\)?