Question 1 (Similar to 17.12)

(a) Let $f$ be a continuous real-valued function defined on all of $\mathbb{R}$. Show that if $f(r) = 0$ for each $r \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

(b) Let $f$ and $g$ be continuous real-values functions defined on $\mathbb{R}$ such that $f(r) = g(r)$ for each rational number $r \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. (Hint: this is a quick consequence of part (a).)

Question 2

Let $f(x) = 0$ for rational numbers and $f(x) = x$ for irrational numbers.

(a) At the beginning of the course, we proved that the rational numbers are dense in the real numbers: for any real numbers $a < b$ there exists a rational number $r \in \mathbb{Q}$ with $a < r < b$. Now prove that for any real numbers $a < b$ there exists an irrational number $y \in \mathbb{I}$ with $a < y < b$.

(b) Show that $f$ is continuous at $x = 0$.

(c) Show that $f$ is discontinuous at every $x$ in $\mathbb{R} \setminus \{0\}$. (Hint: consider the cases $x_0 \in \mathbb{Q}$ and $x_0 \in \mathbb{I}$ separately. Use denseness of $\mathbb{Q}$ in $\mathbb{R}$ and denseness of $\mathbb{I}$ in $\mathbb{R}$ to construct your sequences.)

Question 3 (Similar to 17.14)

Let $f$ be a real-valued function whose domain is a subset of $\mathbb{R}$. Prove that $f$ is continuous at $x_0$ in $\text{dom}(f)$ if and only if for every sequence $x_n$ in $\text{dom}(f) \setminus \{x_0\}$ that converges to $x_0$, we have $\lim_{n \to \infty} f(x_n) = f(x_0)$.

Question 4 (Similar to 18.4)

Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence $x_n$ in $S$ converging to a number $x_0 \notin S$. Show that there exists an unbounded continuous function on $S$.

Question 5 (Similar to 18.5)

(a) Let $f$ and $g$ be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Use the intermediate value theorem to prove that $f(x_0) = g(x_0)$ for at least one $x_0$ in $[a, b]$.

(b) Let $f$ be a continuous function mapping $[0, 1]$ into $[0, 1]$, i.e. $f$ is a function defined on the interval $[0, 1]$ so that $f(x) \in [0, 1]$ for all $x \in [0, 1]$. Use part (a) to show that there exists a point $x_0 \in [0, 1]$ so that $f(x_0) = x_0$. (Such a point is called a fixed point since $f$ maps the point to itself, leaving its location “fixed”.)

Question 6 (Similar to 18.8)

Suppose $f$ is a real-valued continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Use the intermediate value theorem to prove there exists $x$ between $a$ and $b$ such that $f(x) = 0$. 

Question 7 (Similar to 19.2 and 19.3)

Prove that the following functions are uniformly continuous on the indicated set by directly verifying the definition.

(a) \( f(x) = 2x - 1 \) on \( \mathbb{R} \)

(b) \( f(x) = \frac{x}{x-1} \) on \( [0, \frac{1}{2}) \)

Question 8

Let \( f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \)

(a) Prove \( f \) is continuous on \( \mathbb{R} \). (You may assume \( g_1(x) = x \) and \( g_2(x) = \sin(x) \) are continuous on \( \mathbb{R} \).)

(b) Prove \( f \) is uniformly continuous on any bounded subset of \( \mathbb{R} \).

Question 9

Lightning Round!
Consider the functions \( f(x) = \frac{1}{x-1} \) and \( g(x) = e^x \).

(i) What is \( \operatorname{dom}(f \circ g) \)? Does there exist \( x_0 \in (0, +\infty) \) so that \( f \circ g(x_0) = \pi \)?

(ii) Let \( h = fg \). What is \( \operatorname{dom}(h) \)? Is \( h \) bounded on the set \((-1, 0)\)?