Practice Quiz 5 Solutions
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2. Note that since \(0 \leq f(x) \leq x\),
\[0 \leq x_n = f(x_{n-1}) \leq x_{n-1}.\]
Consequently, \(x_n\) is a decreasing sequence that is bounded below by \(0\) and bounded above by \(f(x_0)\).
Since all bounded, monotone sequences converge, \(x_n\) converges to some \(y \in \mathbb{R}\). Furthermore, since \(x_n \geq 0\) for all \(n\), we have \(y_0 \geq 0 \iff y_0 \in [0, +\infty)\).

5. As in HW 2, Q14(a), since \(\lim_{n \to \infty} x_n = y_0\), we also have \(\lim_{n \to \infty} x_{n-1} = y_0\).
Since these sequences lie in \([0, +\infty)\) and \(f\) is cts on \([0, +\infty)\), we have
\[x_n = f(x_{n-1}) \Rightarrow \lim_{n \to \infty} x_n = \lim_{n \to \infty} f(x_{n-1}) = y_0 = f(\lim_{n \to \infty} x_{n-1}) = \lim_{n \to \infty} f(x_{n-1}) = y_0 = f(y_0).\]
Thus, \(y_0\) is a fixed point of \(f\).
(2) Assume for the sake of contradiction that $f$ is not bounded on $S$. Then for all $k \in \mathbb{N}$, there exists $x_k \in S$ s.t. $f(x_k) = k$. In particular, $\lim_{k \to \infty} f(x_k) = +\infty$.

Since $x_k \in S$ for all $k \in \mathbb{N}$ and $S$ is a bounded set, $x_k$ is a bounded sequence. By Bolzano–Weierstrass, it has a convergent subsequence $x_{k_e}$.

Since $f$ is uniformly continuous, $f(x_{k_e})$ is also convergent. However, $f(x_{k_e})$ is also a subsequence of $f(x_k)$, so $\lim_{k \to \infty} f(x_{k_e}) = +\infty$. This is a contradiction. Thus, $f$ must be bounded on $S$.

(b) Since $\log(x)$ is not bounded on $(0,1)$ (in fact, $\lim_{x \to 0^+} \log(x) = -\infty$) and $(0,1)$ is a bounded set, part (a) ensures $\log(x)$ is not uniformly continuous.
3. It suffices to prove there exists \( x \in [0,1] \) so that \( f(x) = f(x+1) \). Define \( g(x) = f(x) - f(x+1) \) for \( x \in [0,1] \). Then \( g(1) = f(1) - f(2) \) and \( g(0) = f(0) - f(1) = f(2) - f(1) \). Thus, \( g(1) = -g(0) \), so either \( g(1) \leq 0 \leq g(0) \) or \( g(0) \leq 0 \leq g(1) \). By the Intermediate Value Theorem, there exists \( x \in [0,1] \) so that \( g(x) = 0 \). This implies there exists \( x \in [0,1] \) so that \( f(x) - f(x+1) = 0 \), which gives the result.

4. First, we show \( f_x \) is decreasing. Suppose \( x_0 \leq x_1 \). Then \( [f(y) : a \leq y \leq x_0] \leq [f(y) : a \leq y \leq x_1] \), so \( f_x(x_0) = \inf [f(y) : a \leq y \leq x_0] \geq \inf [f(y) : a \leq y \leq x_1] = f_x(x_1) \).

Now, we show \( f_x \) is continuous. In fact, we will show \( f_x \) is uniformly continuous on \([a,b]\).

Fix \( \varepsilon > 0 \). Since \( f \) is continuous on \([a,b]\), it is uniformly continuous on \([a,b]\), and there exists \( \delta > 0 \) so that \( x, x' \in [a,b] \) and \( |x - x'| < \delta \) ensures \( |f(x) - f(x')| < \frac{\varepsilon}{2} \).
It suffices to show that $x, x_0 \in [a, b]$ and $|x - x_0| < \delta$ ensures $|f(x) - f(x_0)| < \varepsilon$.

Without loss of generality, suppose $x \leq x_0$.

Then,

$$|f_+(x) - f_+(x_0)|$$

is decreasing

$$= f_+(x) - f_+(x_0)$$

$$= \inf f(y) : a \leq y \leq x_2 - \inf f(y) : a \leq y \leq x_2$$

$$= \inf \{ f(y) : a \leq y \leq x_2 \} - \min \{ \inf f(y) : a \leq y \leq x_2, \inf f(y) : x \leq y \leq x_2 \}$$

$$= \begin{cases} 0 & \text{if } \inf f(y) : a \leq y \leq x_2 \leq \inf f(y) : x \leq y \leq x_2 \\ \inf f(y) : a \leq y \leq x_2 - \inf f(y) : x \leq y \leq x_2 & \text{if } \inf f(y) : a \leq y \leq x_2 \geq \inf f(y) : x \leq y \leq x_2 \end{cases}$$

Suppose we are in the second case and $y \in [x, x_0]$. Then $|x - x_0| < \delta$ implies $|y - x| < \delta$, so $|f(x) - f(y)| < \frac{\varepsilon}{2}$ implies $f(y) > f(x) - \frac{\varepsilon}{2}$. In particular, $\inf f(y) : x \leq y \leq x_2 \geq f(x) - \frac{\varepsilon}{2}$. 

Combining this with the above estimates, we obtain

\[ |f_\varepsilon(x) - f_\varepsilon(x_0)| \leq \begin{cases} 0 \\ \inf \{ f(y) : a \leq y \leq x^3 - f(x) - \frac{\varepsilon}{2} \} \end{cases} \]

\[ \leq 0 + \frac{\varepsilon}{2} \]

\[ < \varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, this shows \( f_\varepsilon \) is uniformly continuous on \([a,b]\).

5) Let \( I = (\frac{a}{2}, \frac{3a}{2}) \). Then \( I \setminus \overline{Ea^3} = \text{dom}(f) \)

and for any sequence \( x_n \in I \setminus \overline{Ea^3} \)

\[ \lim_{n \to \infty} x_n = a, \]

\[ \lim_{n \to \infty} \frac{x_n^2 - a^2}{x_n - a} = \lim_{n \to \infty} \frac{(x_n-a)(x_n+a)}{x_n-a} = \lim_{n \to \infty} x_n + a = 2a. \]

Since \( x_n \) was arbitrary, \( \lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a. \)

6) Similar to 5a. Ask me if you have questions.
Since $L_1 = \lim_{x \to a^-} f(x)$ and $L_2 = \lim_{x \to a^+} f(x)$ exist, there exist intervals $I_1 = (b_1, a)$ and $I_2 = (b_2, a)$ with $I_1 \subseteq \text{dom}(f_1)$, $I_2 \subseteq \text{dom}(f_2)$ and, for every sequence $x_n \in I_1$, $x_n \in I_2$, with $\lim_{n \to \infty} x_n = a$,

\[
L_1 = \lim_{n \to \infty} f_1(x_n), \quad \lim_{n \to \infty} f_2(x_n) = L_2.
\]

Let $b_3 = \max \{b_1, b_2, b_3\}$. Then $b < a$ and if we define $I_3 = (b_3, a)$, then $I_3 \subseteq I_1 \cap I_2$, and for every sequence $x_n \in I_3$ s.t. $\lim_{n \to \infty} x_n = a$, we have

\[
L_1 = \lim_{n \to \infty} f_1(x_n), \quad \lim_{n \to \infty} f_2(x_n) = L_2.
\]

Furthermore, since $f_1(x_n) \geq f_2(x_n)$ for all $n \in \mathbb{N}$, by Practice Quiz 2, Q1, $\lim_{n \to \infty} f_1(x_n) \geq \lim_{n \to \infty} f_2(x_n)$. This shows $L_1 \geq L_2$. 
(b) One cannot conclude $L_1 > L_2$. Consider $f_1(x) = 0$, $f_2(x) = x$, $b = -1$, and $a = 0$. Then for any sequence $x_n$ in $(-1, 0)$ with $\lim_{n \to \infty} x_n = 0$, we have

\[
\lim_{n \to \infty} f_1(x_n) = \lim_{n \to \infty} 0 = 0, \quad \lim_{n \to \infty} f_2(x_n) = \lim_{n \to \infty} x_n = 0.
\]

Thus, $\lim_{x \to 0^-} f_1(x) = 0 = L_1$ and $\lim_{x \to 0^-} f_2(x) = 0 = L_2$ so $L_1 \neq L_2$.

(7)

Since $\lim_{x \to a^-} f_1(x)$ and $\lim_{x \to a^-} f(x)$ exist,

there exist intervals $I_1 = (b_1, a)$ and $I_3 = (b_3, a)$ with $I_1 \subseteq \text{dom}(f_1)$, $I_3 \subseteq \text{dom}(f)$ and, for every sequence $x_n^1 \in I_1$, $x_n^3 \in I_3$ with $\lim_{n \to \infty} x_n^1 = \lim_{n \to \infty} x_n^3 = a$,

\[
L = \lim_{n \to \infty} f_1(x_n^1) = \lim_{n \to \infty} f_3(x_n^3).
\]

Let $b_2 = \max\{b_1, b_3, b_3^2\}$. Then $b < a$. 

and if we define \( I_2 = (b, a) \), we have
\[
I_2 \subseteq (b, a) \cap I_1 \cap I_3 \subseteq \text{dom}(f_1) \cap \text{dom}(f_2) \cap \text{dom}(f_3).
\]
Furthermore, for any sequence \( x_n \in I_2 \) s.t. \( \lim_{n \to \infty} x_n = a \), we have
\[
f_1(x_n) \leq f(x_n) \leq f_3(x_n),
\]
so by the squeeze lemma,
\[
\lim_{n \to \infty} f(x_n) = L.
\]

Since \( x_n \) was arbitrary, this shows
\[
\lim_{x \to a^-} f(x) = L.
\]

\( \square \)

Let \( f(x) = x^2 \). Fix \( x \in S \). In homework 4.06, you proved that any polynomial is continuous on \( \mathbb{R} \). Consider the polynomial \( g(y) = (x - y)^2 \). Since \( \lim_{n \to \infty} \frac{1}{n} = 0 \),
\[ \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} (x - \frac{1}{n})^2 = \lim_{n \to \infty} g\left(\frac{1}{n}\right) = g(0) = x^2 = f(x). \]

Since \( x \in S \) was arbitrary, this shows that \( \lim_{n \to \infty} f_n(x) = f(x) \).

6) Let \( f(x) = x^2 \). Let \( \varepsilon > 0 \), let \( N = \frac{3}{\varepsilon} \). Then \( n > N \) ensured \( \frac{3}{n} < \varepsilon \). Thus,

\[ |f_n(x) - f(x)| = |(x - \frac{1}{n})^2 - x^2| \]

\[ = \left| -\frac{2x}{n} + \frac{1}{n^2} \right| \]

\[ \leq \left| \frac{2x}{n} \right| + \frac{1}{n^2} \]

\[ \leq \frac{2}{n} + \frac{1}{n^2} \quad \text{for} \quad x \in [0, 1] \]

\[ \leq \frac{2}{n} + \frac{1}{n} \]

\[ = \frac{3}{n} \]

\[ < \varepsilon \]

for all \( x \in [0, 1] \). Since \( \varepsilon > 0 \) was arbitrary, this shows \( f_n \to f \) uniformly.
Suppose $f_n \to f$ uniformly on $S$. We seek to show

$$\limsup_{n \to \infty} \{ |f(x) - f_n(x)| : x \in S \} = 0.$$ 

Fix $\varepsilon > 0$. Since $f_n \to f$ uniformly, there exists $N$ s.t. $n > N$ ensures $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$. Thus $n > N$ ensures $\varepsilon$ is an upper bound for $\{ |f(x) - f_n(x)| : x \in S \}$, so

$$\sup \{ |f(x) - f_n(x)| : x \in S \} - 0$$

$$= \sup \{ |f(x) - f_n(x)| : x \in S \} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows

$$\limsup_{n \to \infty} \{ |f(x) - f_n(x)| : x \in S \} = 0.$$

The other direction is similar. Ask me if you have questions.