Math 117: Practice Quiz 5

Question 1
Suppose \( f(x) \) is a continuous function on \([0, +\infty)\) satisfying \( 0 \leq f(x) \leq x \) for all \( x \in [0, +\infty) \).
Fix \( x_0 \in [0, +\infty) \) and define a sequence \( x_n \) recursively as follows: \( x_1 = f(x_0) \) and \( x_n = f(x_{n-1}) \).

(a) Prove that \( x_n \) converges to some \( y_0 \in [0, +\infty) \).

(b) Prove that \( y_0 \) is a fixed point of \( f \), i.e. \( f(y_0) = y_0 \).

Question 2
(a) Suppose \( f \) is uniformly continuous on a bounded set \( S \). Prove that \( f \) is bounded on \( S \).

(Hint: Assume for the sake of contradiction that \( f \) is not bounded on \( S \). Prove that there exists a convergent sequence \( s_k \in S \) so that \( \lim_{k \to +\infty} f(s_k) = +\infty \). Then use Bolzano-Weierstrass and the fact that uniformly continuous functions map convergent sequences to convergent sequences.)

(b) Prove that \( f(x) = \log(x) \) is not uniformly continuous on \((0, 1)\).

Question 3
Suppose \( f \) is continuous on \([0, 2] \) and \( f(0) = f(2) \). Use the intermediate value theorem to prove that there exist \( x, y \in [0, 2] \) such that \( |y - x| = 1 \) and \( f(x) = f(y) \).

Question 4 (similar to 19.12)
Let \( f \) be a continuous function on \([a, b] \). Show that the function \( f_* \) defined by
\[
f_*(x) = \inf \{ f(y) : a \leq y \leq x \}, \quad \text{for } x \in [a, b]
\]
is a decreasing continuous function on \([a, b] \). (Recall that a function \( f \) is \emph{decreasing} if \( x \leq y \) implies \( f(x) \geq f(y) \).)

Question 5 (similar to 20.11)
Find the following limits. Justify your answers using the definition of a two sided limit.

(a) \( \lim_{x \to a} \frac{x^2 - a^2}{x - a} \)

(b) \( \lim_{x \to b} \frac{\sqrt{x} - \sqrt{b}}{x - b}, \quad b > 0 \)

(c) \( \lim_{x \to a} \frac{x^3 - a^3}{x - a} \) (Hint: \( x^3 - a^3 = (x - a)(x^2 + ax + a^2) \).)
Question 6 (similar to 20.16)

Suppose the limits $L_1 = \lim_{x \to a^{-}} f_1(x)$ and $L_2 = \lim_{x \to a^{-}} f_2(x)$ exist.

1. Show if $f_1(x) \geq f_2(x)$ for all $x$ in some interval $(b, a)$, then $L_1 \geq L_2$.

2. Suppose that, in fact, $f_1(x) > f_2(x)$ for all $x$ in some interval $(b, a)$. Can you conclude $L_1 > L_2$? Justify your answer with a proof or a counterexample.

Question 7 (similar to 20.17)

Consider functions $f_1$, $f_2$, and $f_3$ on $(b, a)$ satisfying

$$f_1(x) \leq f_2(x) \leq f_3(x).$$

Suppose that $\lim_{x \to a^{-}} f_1(x) = \lim_{x \to a^{-}} f_3(x) = L$. Prove that $\lim_{x \to a^{-}} f_2(x) = L$.

(Note that this is not an immediate consequence of Q6, since you must first prove that $\lim_{x \to a^{-}} f_2(x)$ exists.)

Question 8

Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0, 1]$.

(a) Does the sequence $f_n$ converge pointwise on $[0, 1]$? Justify your answer with a proof, using the definition of pointwise convergence.

(b) Does the sequence $f_n$ converge uniformly on $[0, 1]$? Justify your answer with a proof, using the definition of uniform convergence.

Question 9

Prove that a sequence of functions $f_n$ on a set $S \subseteq \mathbb{R}$ converges to a function $f$ uniformly on $S$ if and only if

$$\lim_{n \to +\infty} \sup \{|f(x) - f_n(x)| : x \in S\} = 0.$$