Math 117: Homework 1
Due Monday, April 4th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.
Submit your answers at https://www.gradescope.com/courses/381472.

Question 1 (Similar to 1.2 and 1.5)

(a) Prove by induction that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in \mathbb{N}$.
(b) Prove by induction that $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

Question 2* (Similar to 1.8)

We now describe a useful extension of the principle of mathematical induction. Given a list of propositions $P_m, P_{m+1}, \ldots$, if you can show that

(i) $P_m$ is true;
(ii) for all $n \geq m$, if $P_n$ is true then $P_{n+1}$ is true;

then $P_m, P_{m+1}, P_{m+2} \ldots$ are all true.

Use this extension of the principle of mathematical induction to prove the following statements:

(a) $n^2 > n + 1$ for all integers $n \geq 2$.
(b) $n! > n^2$ for all integers $n \geq 4$. (Recall that $n! = n(n-1)\cdots2\cdot1$. For example, $5! = 5\cdot4\cdot3\cdot2\cdot1$.)

Question 3* (Similar to 3.8)

Suppose $p, q \in \mathbb{Q}$. Suppose that for all $s \in \mathbb{Q}$ with $s > p$, we have $q \leq s$. Prove that $q \leq p$.
(Hint: Prove the result by contradiction, using the fact that, between any two rational numbers, there is a rational number.)

Question 4

Using the ordered set properties (i)-(vi) listed in class, prove the following statements for all $a, b \in \mathbb{Q}$. You may use the fact that $0 = -0$ and $0 \cdot a = 0$ for all $a \in \mathbb{Q}$.

(i) If $a \leq b$, then $-b \leq -a$.
(ii) $0 \leq a^2$.

Question 5 (Question 1 from Practice Quiz) (Similar to 3.1)

(a) Why is $\mathbb{N}$ not a field?
(b) Why is $\mathbb{Z}$ not a field?
Question 6* (Similar to 3.4)

The textbook proves the following result (see Theorem 3.2, p16):

**THEOREM 1.** Suppose $F$ is an ordered field. Then the following properties hold for all $a, b, c \in F$:

(i) if $a \leq b$, then $-b \leq -a$;
(ii) if $a \leq b$ and $c \leq 0$, then $bc \leq ac$;
(iii) if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$;
(iv) $0 \leq a^2$, where $a^2$ is an abbreviation for $a \cdot a$;
(v) if $0 < a$, then $0 < 1/a$.

Using this theorem and the definition of an ordered field, prove the following results for an ordered field $F$. Explain which properties of a field you use (A1-A4, M1-M4, DL), which properties of an ordered field you use (O1-O5), and which parts of Theorem 1 you use (i-v).

(a) For 0 and 1 as in the definition of a field, $0 < 1$. (Recall that $a < b$ means $a \leq b$ and $a \neq b$.)

(b) For all $a, b \in F$, if $0 < a < b$, then $0 < 1/b < 1/a$.

Question 7*

Use the definition of the absolute value to prove the following results for all $a, b \in \mathbb{R}$. You may use all facts you learned about $\mathbb{R}$ in previous courses without further justification. (In particular, you do not need to point out where you use the properties of an ordered field.)

(i) $|a| \geq 0$
(ii) $|ab| = |a| \cdot |b|
(iii) $|a| \geq a$ and $|a| \geq -a$
(iv) $|a + b| \leq |a| + |b|

Question 8 (Similar to 3.5)

Use the definition of the absolute value to prove the following results for all $a, b \in \mathbb{R}$. You may use all facts you learned about $\mathbb{R}$ in previous courses without further justification. (In particular, you do not need to point out where you use the properties of an ordered field.)

(a) For any $a, b \in \mathbb{R}$, prove that $|b| \leq a$ if and only if $-a \leq b \leq a$.

(b) For any $a, b \in \mathbb{R}$, prove that $||a| - |b|| \leq |a - b|$. This is known as the reverse triangle inequality.

Question 9 (Similar to 3.7)

Use the definition of the absolute value to prove the following results for any $a, b, c \in \mathbb{R}$. You may use all facts you learned about $\mathbb{R}$ in previous courses without further justification. We will use these inequalities repeatedly throughout the course.

(a) Prove that $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$

(b) Prove that $|a - b| < c$ if and only if $b - c < a < b + c$.

**Extra credit:** For each part, draw a picture of the real number line to illustrate the statement that you proved.
Question 10* (Question 5 from practice quiz)

In lecture, we claimed that the notion of supremum is a generalization of the notion of maximum. It turns out that if the maximum of a set exists, then the supremum is the maximum. Prove this by showing that if $S$ is a nonempty subset of $\mathbb{R}$ with maximum $s_0$, then $\text{sup}(S) = s_0$. 