Math 117: Homework 2
Due Monday, April 11th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1* (Similar to 3.6)
Prove the following results using the definition of $|·|$ and the triangle inequality.
(a) For any $d, e, f \in \mathbb{R}$, prove that $|d + e + f| \leq |d| + |e| + |f|$.
(b) Prove by induction that for $n \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n \in \mathbb{R}$,
\[|y_1 + y_2 + \cdots + y_n| \leq |y_1| + |y_2| + \cdots + |y_n|.

Question 2* (Similar to 4.5)
Let $T$ be a nonempty subset of $\mathbb{R}$ that is bounded above.
(a) Prove that $\sup T \in T$ implies $\sup T = \max T$.
(b) Prove that if $\sup T$ is not an element of $T$, then the maximum of $T$ does not exist.

Question 3 (Similar to 4.6)
Let $S$ be a nonempty bounded subset of $\mathbb{R}$.
(a) Prove that $\inf S \leq \sup S$.
(b) What can you say about the number of elements in $S$ if $\inf S = \sup S$? Justify your answer.

Question 4* (Similar to 4.7)
Suppose $A$ and $B$ are nonempty bounded subsets of $\mathbb{R}$.
(a) Prove that if $B \supseteq A$, then $\inf B \leq \inf A \leq \sup A \leq \sup B$.
(b) Prove $\sup(A \cup B) = \max\{\sup A, \sup B\}$. (Note: for this part, do not assume $A \subseteq B$.)

Question 5 (Similar to 4.9)
Consider the following proposition:

PROPOSITION 1. Every nonempty subset $S$ of $\mathbb{R}$ that is bounded below has an infimum.
(See also Corollary 4.5 on p23 of the textbook.)
This question will lead you through the proof of the proposition.
(a) Suppose $S$ is as in the proposition above. Define the set $-S = \{-s : s \in S\}$. Show that $-S$ is bounded above.
(b) Use the definition of $\mathbb{R}$ to prove that $-S$ has a supremum, $\sup(-S)$.
(c) Prove that $-\sup(-S)$ is the infimum of $S$. 
Question 6 (Similar to 4.14)

Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$, and define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Prove $\sup(A + B) = \sup A + \sup B$.

**Hint:** Show that for all $b \in B$, $\sup(A + B) - b$ is an upper bound for $A$. Hence $\sup A \leq \sup(A + B) - b$ for all $b \in B$. Then show $\sup(A + B) - \sup A$ is an upper bound for $B$.

Question 7* (Similar to 4.8)

Let $A$ and $B$ be nonempty subsets of $\mathbb{R}$ with the following property: $a \leq b$ for all $a \in A$ and $b \in B$.

(a) Show that $A$ is bounded above and $B$ is bounded below.

(b) Prove $\sup A \leq \inf B$.

(c) Given an example of $A$ and $B$ satisfying the above property where $A \cap B$ is nonempty.

(d) Give an example of $A$ and $B$ satisfying the above property where $\sup A = \inf B$ and $A \cap B$ is the empty set. You do not need to justify your example with a proof.

Question 8* (Similar to 4.1-4.4)

For each of the sets below, answer the following questions: Is it bounded above? If so, what is its supremum? Is it bounded below? If so, what is its infimum? You do not need to justify your answers.

(a) $[-\sqrt{2}, \sqrt{2}]$

(b) $\{-1, 0, e, \pi\}$

(c) $\{1\}$

(d) $\bigcup_{n=1}^{\infty}[2n - 1, 2n)$

(e) $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$

(f) $\{x \in \mathbb{R} : x^2 < 1\}$

(g) $\bigcap_{n=1}^{\infty}(-1 - \frac{1}{n}, 1 + \frac{1}{n})$

Question 9 (Similar to 4.1-4.4)

Follow the same instructions as in Question 1 for the following sets:

(a) $\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}$

(b) $\left\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

(c) $\{q \in \mathbb{Q} : q \geq 0\}$

(d) $\{q \in \mathbb{Q} : q^2 \geq 0\}$
(e) $\cap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right)$

(f) $\{x \in \mathbb{R} : x^3 \geq 2\}$

(g) $\{\sin(n\pi) : n \in \mathbb{N}\}$

**Question 10***

Given $s,t \in \mathbb{R}$, consider the set $(s,t]$. Find the maximum, minimum, supremum, and infimum of the set or state that they do not exist. Justify your answers with proofs.

**Question 11*** (Similar to 4.10)

Prove that if $s > 0$, then there exists $n \in \mathbb{N}$ satisfying $\frac{1}{n} < s < n$.

**Question 12** (Similar to 4.15)

Let $a,b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$. (Compare to Question 3 on HW 1.)