Homework 4 Solutions
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1. Assume for the sake of contradiction that $a > b$. Define $\varepsilon := a - b$. Then $\varepsilon > 0$ and $b + \varepsilon = a$. Thus, there exists $\varepsilon > 0$ s.t. $a \geq b + \varepsilon$. This is a contradiction.

2. (a) By definition $s_{n+1} = s_n + \frac{d_{n+1}}{10^{n+1}}$. Since $d_{n+1} \geq 0$, $s_{n+1} \geq s_n$.

(b) Taking $a = \frac{1}{10}$ in Q4(a) gives

$$1 + \frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^n} = \frac{1 - (\frac{1}{10})^{n+1}}{1 - \frac{1}{10}}$$

$$\Leftrightarrow 9 \cdot \frac{1}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 10 - (\frac{1}{10})^n$$

$$\Leftrightarrow \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - (\frac{1}{10})^n$$

(c) Since $s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$ and $d_i \leq 9$ for all $i = 1, \ldots, n$,

$s_n = K + \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = K + \frac{1}{10} - \frac{1}{10^n} \leq K + 1$.

Therefore $s_n$ is bounded above. Since $s_n \geq 0$, it is also bounded below, hence bounded.
(a) Let $s_n = .99\ldots 9$. Then $s_n = 1 - \frac{1}{10^n}$. Since $\lim_{n \to \infty} \frac{1}{10^n} = 0$, $\lim_{n \to \infty} \frac{1}{10^n + 1} = \frac{1}{10^n + 1} = 0$. Hence $\lim_{n \to \infty} 1 - \frac{1}{10^n + 1} = 0$. Thus, $\bar{q} = \lim_{n \to \infty} s_n = 1$.

(b) Assume $s_n$ is a bounded sequence. Then there exists $M_0$ s.t. $|s_n| \leq M_0$ for all $n \in \mathbb{N}$. Hence $\sup \{ |s_n| : n > N \} \leq M_0$ for all $N \in \mathbb{N}$. By HW3, Q9(b), this implies $\limsup_{n \to \infty} s_n \leq M_0$, so $\limsup_{n \to \infty} |s_n| \leq M_0 < +\infty$.

Now, assume $\limsup_{n \to \infty} |s_n| < +\infty$. Let $M_0 = \limsup_{n \to \infty} |s_n| = \lim_{n \to \infty} \sup \{ |s_n| : n > N \}$, and $\alpha = \lim_{n \to \infty} |s_n|$. Since $a_n$ is a convergent sequence, it is bounded, and there exists $M_0$ s.t. $|a_n| \leq M_0$ for all $n \in \mathbb{N}$. In particular, $|a_1| \leq M_0$ implies $|\sup \{ |s_n| : n > 1 \}| \leq M_0$, so $|s_n| \leq \max \{ |s_1|, M_0 \}$. Thus $s_n$ is a bounded sequence.
4. \(a\) Base case: When \(m=0\), \(1 = \frac{1-a}{1-a}\)

Inductive step: Suppose \(1 + a + a^2 + \cdots + a^{m-1} = \frac{1-a^m}{1-a}\).

Then \(1 + a + a^2 + \cdots + a^m = \frac{1-a^m}{1-a} + a^m = \frac{1-a^m + a^m}{1-a} = \frac{1-a^{m+1}}{1-a}\),

which completes the proof.

\(b\) By the hint and part \(a\),

\[
\sum_{i=0}^{m-1} a^i = \sum_{i=0}^{m-1} a^i - \sum_{i=0}^{n-1} a^i = \frac{1-a^m}{1-a} - \frac{1-a^n}{1-a} = \frac{a^n-a^m}{1-a}
\]

\(c\) Note that, for \(m > n\),

\[
|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n|
\]

\[
\leq |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n|
\]

\[
\leq 4^{-(m-1)} + 4^{-(m-2)} + \cdots + 4^{-n}
\]

\[
= \frac{4^{-1}}{4} - \frac{4^{-n}}{4}
\]

Furthermore, for all \(\varepsilon > 0\),

\[
\frac{4}{3} \left(\frac{1}{4}\right)^n < \varepsilon \Leftrightarrow \left(\frac{1}{4}\right)^n < \frac{3\varepsilon}{4} \Leftrightarrow n \log\left(\frac{1}{4}\right) < \log\left(\frac{3\varepsilon}{4}\right)
\]

\[
\Rightarrow n > \frac{\log\left(\frac{3\varepsilon}{4}\right)}{\log\left(\frac{1}{4}\right)}.
\]

Let \(\varepsilon > 0\). Define \(N = \frac{\log\left(\frac{3\varepsilon}{4}\right)}{\log\left(\frac{1}{4}\right)}\). Then \(m, n > N\) ensures \(|s_m - s_n| < \varepsilon\).

Therefore \(s_n\) is Cauchy.
Yes. The sequence $s_n$ converges since all Cauchy sequences are convergent.

$\textcircled{5a} \ s_n = (3, 2, 3, 4, 3, 2, 3, 4, \ldots)$

$\textcircled{5b} \ (3, 3, 3, \ldots)$

$\textcircled{6a} \ False. \ Consider: \ s_n = (-1)^n 2.$

Then $\limsup_{n \to \infty} s_n = \lim \sup \{s_n: n \geq N\}$

$= \lim_{n \to \infty} 2 = 2.$

However, all odd elements of $s_n$ are strictly less than 1.99.

$\textcircled{6b} \ False. \ Consider \ s_n = b + \frac{1}{n}.$

Since $s_n$ is convergent,

$\lim_{n \to \infty} s_n = b = \lim_{n \to \infty} \limsup_{n \to \infty} s_n.$

However $s_n > b$ for all $n.$
A sequence \( s_n \) converges to a limit \( s \) if for all \( \varepsilon > 0 \), \( \exists N \text{ s.t. } n > N \) ensures \( |s_n - s| < \varepsilon \).

A sequence \( s_n \) doesn't converge to a limit \( s \) if \( \exists \varepsilon > 0 \) s.t. \( \forall N, \exists n > N \text{ s.t. } |s_n - s| \geq \varepsilon \).

We construct such a subsequence.
Taking \( N = 1 \) in part \( \circled{6} \), \( \exists n_1 > 1 \) s.t. \( |s_{n_1} - s| \geq \varepsilon \). Suppose we have chosen \( n_k \). Taking \( N = n_{k-1} \) in part \( \circled{6} \), \( \exists n_k > n_{k-1} \text{ s.t. } |s_{n_k} - s| \geq \varepsilon \).

Therefore, there exists a subsequence \( s_{n_k} \) s.t. \( |s_{n_k} - s| \geq \varepsilon \) \( \forall k \).

If \( \lim s_n = s \), then all subsequences of \( s_n \) also converge to \( s \). Hence every subsequence \( s_{n_k} \) has a further subsequence \( s_{n_{k\circ}} = s_{n_k} \) that converges to \( s \).
Suppose \( \lim s_n = s \). Then, \( \exists \varepsilon > 0 \) s.t. \( \forall N, \exists n > N \) s.t. \( |s_n - s| \geq \varepsilon \). We now construct the subsequence described in the hint. First, taking \( N = 1 \), we have \( \exists n_1 > 1 \) s.t. \( |s_{n_1} - s| \geq \varepsilon \). Suppose we have chosen \( n_k \). Taking \( N = n_{k-1} \), we see that \( \exists n_k > n_{k-1} \) s.t. \( |s_{n_k} - s| \geq \varepsilon \).

Therefore there exists a subsequence \( s_{n_k} \) s.t. \( |s_{n_k} - s| \geq \varepsilon \) \( \forall k \). Since \( s_{n_k} \) is always at least distance \( \varepsilon \) from \( s \), no further subsequence of \( s_{n_k} \) can converge to \( s \).

(i) The limit exists. By the Basic Examples theorem, the limit is zero.

(ii) Since \( b_n \) converges to zero, all subsequences also converge to zero. Thus, the set of subsequential limits is \( \mathbb{R} \).
(iii) Since the limit exists, 
\[ \lim_{n \to \infty} b_n = \limsup_{n \to \infty} b_n = \lim_{n \to \infty} b_n = 0. \]

(i) True. If a sequence is unbounded, it must either be unbounded above or unbounded below.

As proved in class, if a sequence is unbounded above, \(+\infty\) is a subsequential limit, and if a sequence is unbounded below, \(-\infty\) is a subsequential limit.

(ii) False. Consider \( s_n = (-1)^n \). Then \(|s_n| \leq 1\) for all \( n \in \mathbb{N} \), so \( s_n \) is a bounded sequence. However, \((1, 1, 1, 1, \ldots)\) and \((-1, -1, -1, -1, \ldots)\) are subsequences, so \( 1 \) and \(-1\) both belong to the set of subsequential limits.

(iii) Define \( x_n = \frac{\sqrt{2}}{n} \). As shown in class, \( \sqrt{2} \) is an irrational number. Since \( \mathbb{Q} \) is a field, the product of two rational
numbers is a rational number. Since \( N \in \mathbb{Q} \) and \( x_n \cdot n = \sqrt{2} \in \mathbb{R} \), we must have that \( x_n \not\in \mathbb{Q} \), so \( x_n \) is a sequence of irrational numbers.

Claim: \( \lim_{n \to \infty} x_n = 0 \). We must show that for all \( \varepsilon > 0 \), there exists \( N \) s.t. \( n > N \) ensures \( |x_n| < \varepsilon \). Note that

\[
|x_n| = \left| \frac{x_n}{n} \right| = \frac{\sqrt{2}}{n} < \varepsilon \iff \sqrt{2} < n.
\]

Therefore, for all \( \varepsilon > 0 \), if we take \( N = \frac{\sqrt{2}}{\varepsilon} \), then for all \( n > N \), \( |x_n| < \varepsilon \).

(b) Define \( r_n = \overline{1.41421...} \), the first \( n \) digits of decimal approximation of \( \sqrt{2} \).

Or more precisely, we define \( r_n \) by \( r_n = \left\lfloor \sqrt{2} \cdot 10^n \right\rfloor / 10^n \), where \( \left\lfloor a \right\rfloor \) represents the largest integer less than or equal to \( a \). Then \( r_n \in \mathbb{Q} \).

Claim: \( \lim_{n \to \infty} r_n = \sqrt{2} \). Note that

\[
|r_n - \sqrt{2}| = 10^{-n} \left| \sqrt{2} \cdot 10^n - \sqrt{2} \cdot 10^n \right| \leq 10^{-n},
\]

and \( 10^{-n} < \varepsilon \iff \frac{1}{\varepsilon} < 10^n \iff \log_{10} \frac{1}{\varepsilon} < n. \)
Therefore, for all \( \varepsilon > 0 \), if we take 
\[ N = \log_{10} \frac{1}{\varepsilon}, \]
then for all \( n > N \), 
\[ |r_n - \frac{1}{2}| < \varepsilon. \]

12. First, note that \( \liminf S_n \leq \limsup S_n \) by definition of \( \liminf \) and \( \limsup \).

We now show \( \liminf S_n \leq \liminf S_n \) by first proving the hint.

Note that if \( n > M > N \)
\[ S_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n) \]
\[ \geq \frac{1}{n}(s_{n+1} + \ldots + s_M + \ldots + s_n) \]
\[ \geq \frac{1}{n}(n-N) \sup \{ s_n : n > N \} \]
\[ = \left( 1 - \frac{N}{n} \right) \inf \{ s_n : n > N \} \]
\[ \geq \left( 1 - \frac{M}{n} \right) \inf \{ s_n : n > N \} \]

Therefore \( \left( 1 - \frac{N}{n} \right) \inf \{ s_n : n > N \} \) is a lower bound for the set \( \{ s_n : n > N \} \).

Hence \( \inf \{ s_n : n > M \} \geq \left( 1 - \frac{N}{n} \right) \inf \{ s_n : n > N \} \).
First suppose $N$ is fixed. Since $B_m \geq (1 - \frac{1}{m})B_N$ for all $M > N$, sending $M \to \infty$ gives
$$\limsup_{M \to \infty} B_m = \lim_{M \to \infty} B_m = B_N.$$  
Now, sending $N \to \infty$ gives 
$$\limsup_{N \to \infty} B_m = \lim_{N \to \infty} B_m = \liminf_{N \to \infty} B_m,$$  
which proves the first inequality.

Now we show $\limsup_{m \to \infty} \leq \limsup_{m \to \infty}$. by proving the other hint.

Note that if $n > M > N$,
$$s_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n + s_{n+1} + \ldots + s_M + \ldots + s_n)$$
$$= \frac{1}{n}(s_1 + s_2 + \ldots + s_n) + \frac{1}{n}(s_{n+1} + \ldots + s_M + \ldots + s_n)$$
$$\leq \frac{1}{n}(s_1 + s_2 + \ldots + s_n) + \frac{1}{n}(n-N)\sup \{s_n : n > N\}$$
$$= \frac{1}{n}(s_1 + s_2 + \ldots + s_n) + \sup \{s_n : n > N\}$$
$$\leq \frac{1}{n}(s_1 + s_2 + \ldots + s_n) + \sup \{s_n : n > N\}$$
$$= \frac{1}{n}(s_1 + s_2 + \ldots + s_n) + \sup \{s_n : n > N\}$$

Thus
$$\sup \{s_n : n > M\} \leq \frac{1}{n}(s_1 + s_2 + \ldots + s_n) + \sup \{s_n : n > N\}.$$  

Sending $M \to \infty$ for fixed $N$ gives, 
$$\limsup_{m \to \infty} s_m = \lim_{m \to \infty} A_m \leq 0 + \alpha N.$$  
Then sending $N \to \infty$ gives 
$$\limsup_{m \to \infty} s_m \leq \lim_{N \to \infty} \alpha N = \limsup_{N \to \infty}.$$
which completes the proof.

6) If \( \lim s_n \) exists, then \( \limsup s_n = \liminf s_n \). Hence, by part 4, \( \limsup s_n = \liminf s_n \). Therefore \( \lim s_n \) exists.

7) Consider \( s_n = (-1)^{n+1} \) so \( \lim s_n \) doesn't exist. Then \( s_n = \frac{1}{n} \) for \( n \) odd, \( 0 \) for \( n \) even, so \( \lim s_n = 0 \).