Math 117: Homework 5
Due Monday, May 30th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

**Question 1**

Let \( f(x) = \sqrt{2 + 2x} \) and \( g(x) = e^x \).

(a) Give the domains of \( f + g \), \( fg \), \( f \circ g \), and \( g \circ f \).

(b) Are the functions \( f \circ g \) and \( g \circ f \) equal?

(c) Are the expressions \( f \circ g(-2) \) and \( g \circ f(-2) \) meaningful?

**Question 2**

(a) Prove that, for any \( c \in \mathbb{R} \), the constant function \( f(x) = c \) is continuous.

(b) Consider the function

\[
 f(x) = \begin{cases} 
 1/x & \text{for } x \neq 0 \\
 0 & \text{for } x = 0.
\end{cases}
\]

(i) Prove that \( f(x) \) is not continuous by showing that it violates the sequences definition of continuity.

(ii) Prove that \( f(x) \) is not continuous by showing that it violates the \( \epsilon - \delta \) definition of continuity.

**Question 3**

You may assume that the following functions are continuous on their domains: \( \sin(x), \cos(x), e^x, 2^x, \log(x) \) for \( x > 0 \), and \( x^p \) for \( x > 0 \), where \( p \) is any real number. (We use \( \log(x) \) to denote the natural logarithm.) You may also assume that the constant function \( f(x) = c \) is continuous for any \( c \in \mathbb{R} \).

For the following functions, state the domain of each function and prove that the function is continuous on its domain.

(a) \( f(x) = \cos(1 - (\log(x))^2) \)

(b) \( g(x) = \frac{1}{x^2} \cos \left( \frac{1}{1-x} \right) \) for \( x \neq 0, 1 \)

**Question 4**

(a) Use the sequences definition of continuity to prove that, for any \( k \in \mathbb{R} \), \( g(x) = kx \) is continuous.

(b) Use the \( \epsilon - \delta \) definition of continuity to prove that \( f(x) = |x| \) is a continuous function on \( \mathbb{R} \).

(Hint: use the reverse triangle inequality: for any \( a, b \in \mathbb{R} \), \( ||a| - |b|| \leq |a - b| \).)

(c) Use part (b) and the theorem about composition of continuous functions to prove that if \( g \) is continuous at \( x_0 \in \text{dom}(g) \), then \( |g| \) is continuous at \( x_0 \).
Question 5*

(a) Prove by induction that, if \( n \in \mathbb{N} \), the function \( f(x) = x^n \) is continuous on \( \mathbb{R} \).

(b) Prove by induction that for any real numbers \( a_0, a_1, \ldots, a_n \), the corresponding polynomial function \( p(x) = a_0 + a_1x + \cdots + a_nx^n \) is continuous on \( \mathbb{R} \). (Hint: use part (a).)

(c) A rational function \( f \) is a function of the form \( p/q \) where \( p \) and \( q \) are polynomial functions, as in part (b). The domain of \( f \) is \( \{ x \in \mathbb{R} : q(x) \neq 0 \} \). Prove that every rational function is continuous on its domain. (Hint: use part (b).)

Question 6*

Let \( f \) and \( g \) be real-valued functions.

(a) Prove that \( \min(f,g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g| \).

(b) Combine part (a) with Q4(c) to prove that, if \( f \) and \( g \) are continuous, then \( \min(f,g) \) is continuous.

Question 7

Prove that the following functions are discontinuous at \( x_0 = 0 \):

(a) \( f(x) = \begin{cases} x + 1 & \text{if } x > 0 \\ x - 1 & \text{if } x \leq 0 \end{cases} \)

(b) \( g(x) = \begin{cases} \cos\left(\frac{x}{x}\right) & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \)

(c) \( \text{sgn}(x) = \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \) (This function is known as the signum function.)

Question 8*

Let \( f \) be a real-valued function whose domain is a subset of \( \mathbb{R} \). Prove that \( f \) is continuous at \( x_0 \) in \( \text{dom}(f) \) if and only if for every sequence \( x_n \) in \( \text{dom}(f) \setminus \{x_0\} \) that converges to \( x_0 \), we have \( \lim_{n \to \infty} f(x_n) = f(x_0) \).

Question 9

Let \( S \subseteq \mathbb{R} \) and suppose there exists a sequence \( x_n \) in \( S \) converging to a number \( x_0 \notin S \). Show that there exists an unbounded continuous function on \( S \). (Hint: Can you think of a function involving \( x - x_0 \)?)
Question 10*

(a) Let $f$ and $g$ be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Use the intermediate value theorem to prove that $f(x_0) = g(x_0)$ for at least one $x_0$ in $[a, b]$. (Hint: apply the IVT to $h(x) = f(x) - g(x)$.)

(b) Let $f$ be a continuous function mapping $[0, 1]$ into $[0, 1]$, i.e. $f$ is a function defined on the interval $[0, 1]$ so that $f(x) \in [0, 1]$ for all $x \in [0, 1]$. Use part (a) to show that there exists a point $x_0 \in [0, 1]$ so that $f(x_0) = x_0$. (Such a point is called a fixed point since $f$ maps the point to itself, leaving its location “fixed”.) (Hint: Again, you want to cook up a new function and apply the IVT.)

Question 11

Suppose $f$ is a real-valued continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Use the intermediate value theorem to prove there exists $x$ between $a$ and $b$ such that $f(x) = 0$.

Question 12*

Consider the set

$$S = \{ \sqrt{5} \ r : r \in \mathbb{Q} \}.$$

(a) Prove that for all $a, b \in \mathbb{R}$ with $a < b$, there exists $s \in S$ satisfying $a < s < b$.

(b) Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \in S^c, \\ 1 & \text{if } x \in S. \end{cases}$$

For all $x_0 \in S^c$, prove that $f$ is discontinuous at $x_0$.

Question 13

Let $f$ be a real-valued function with $\text{dom}(f) \subseteq \mathbb{R}$. Prove that the following are equivalent:

(i) $f$ is continuous at $x_0$

(ii) for every monotonic sequence $x_n$ in $\text{dom}(f)$ that converges to $x_0$, we have $\lim f(x_n) = f(x_0)$.

(Hint: To prove (ii) implies (i), prove the contrapositive, using the result from Homework 4, Q7(c) and the fact that every sequence has a monotonic subsequence.)

Question 14*

One can show that the set of rational numbers $\mathbb{Q}$ can be listed as a sequence $(r_n)$. The exact procedure is a little tedious, but you can get an idea of how it works by considering the following diagram from the textbook: For example, $r_1 = 0, r_2 = 1, r_3 = 1/2$, and so on. Note that some numbers, such as $-1$, are included multiples times. In this way, you can construct a sequence $(r_n)$ that includes all rational numbers (with repetitions).

(a) For any $\epsilon > 0$ and $a \in \mathbb{R}$, show that the set $\{ r \in \mathbb{Q} : |r - a| < \epsilon \}$ contains infinitely many elements. (Hint: Use denseness of the rationals.)
(b) Let \((r_n)\) be the sequence of rational numbers. Use part (a) to show that for any \(a \in \mathbb{R}\), there exists a subsequence \((r_{n_k})\) that converges to \(a\). **(Hint:** Use part (a) to show that the set \(\{n \in \mathbb{N} : |r_n - a| < \epsilon\}\) is infinite.)

(c) Let \((r_n)\) be the sequence of rational numbers. Show that there exists a subsequence \((r_{n_k})\) satisfying \(\lim_{k \to +\infty} r_{n_k} = +\infty\).