Recall:

**MAJOR RESULT #1**

**Thm (Archimedean Property):** If \( a, b \in \mathbb{R} \) satisfy \( a > 0 \) and \( b > 0 \), then there exists \( n \in \mathbb{N} \) so that \( na > b \).

**MAJOR RESULT #2**

**Thm (Q is dense in \( \mathbb{R} \)):** If \( a, b \in \mathbb{R} \) with \( a < b \), there exists \( r \in \mathbb{Q} \) s.t. \( a < r < b \).

We also defined sup(\( S \)) and inf(\( S \)) even when \( S \) is unbounded above or below.

Given \( S \subseteq \mathbb{R} \) nonempty,

\[
\sup(S) = \begin{cases} 
\text{least upper bound of } S \\
+\infty
\end{cases} \quad \text{if } S \text{ is bounded above}
\]

\[
\inf(S) = \begin{cases} 
\text{greatest lower bound of } S \\
-\infty
\end{cases} \quad \text{if } S \text{ is bounded below}
\]
Ch 2: Sequences

Recall: functions

**Def (sequence):** A **sequence** is a function whose domain is a set of the form \(\mathbb{Z}, m, m+1, m+2, \ldots, 3\) for some \(m \in \mathbb{Z}\). We will study sequences whose range is \(\mathbb{R}\).

Typically, the domain of a sequence will be either \(\{0, 1, 2, 3, \ldots, 3\}\) or \(\{1, 2, 3, \ldots, 3\}\).

**Remark:**
To emphasize that a sequence is a special type of function...

Instead of writing \(f(n)\), we write \(S_n\).

We'll often specify a sequence by listing its values in order, \((s_1, s_2, s_3, \ldots)\).
Ex: If \( s_n = \frac{1}{n} \) for \( n \geq 1 \), the sequence is 
\[ (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \]
* If \( s_n = (-1)^n \) for \( n \geq 0 \), the sequence is 
\[ (1, -1, 1, -1, \ldots) \]

Heuristically, a sequence "converges" to some limit \( s \in \mathbb{R} \) if the values of \( s_n \) stay close to \( s \) for large \( n \).

Ex: We expect \( s_n = \frac{1}{n} \) converges to 0.

We expect \( s_n = (-1)^n \) doesn't converge.
Def (convergence):
• A sequence \( S_n \) of real numbers converges to some \( s \in \mathbb{R} \) provided that 
  \[ \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n > N \Rightarrow |S_n - s| < \varepsilon. \]

• The number \( s \) is the limit of \( S_n \), and we write \( \lim_{n \to \infty} S_n = s \) or \( S_n \to s \).

• A sequence that does not converge to any \( s \in \mathbb{R} \) it is said to diverge.

Remark:
• Recall: \( |b| < a \iff -a < b < a \)
• Thus \( |s_n - s| < \varepsilon \iff -\varepsilon < s_n - s < \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon \)
• \( N \) can depend on \( \varepsilon \).
Assume domain of sequence is $\mathbb{N} = \{1, 2, 3, \ldots \}$ unless otherwise specified.

Ex: Consider the sequence $s_n = \frac{1}{n^2}$. We expect that $\lim_{n \to \infty} \frac{1}{n^2} = 0$. Let's prove this!

**Scratchwork:**

\[ |\frac{1}{n^2} - 0| < \varepsilon \iff \frac{1}{n^2} < \varepsilon \iff \frac{1}{\sqrt{\varepsilon}} < n \]

**Proof:** Fix arbitrary $\varepsilon > 0$. Let $N = \frac{1}{\sqrt{\varepsilon}}$. Then for $n > N$, we have

\[ n > \frac{1}{\sqrt{\varepsilon}} \iff \frac{1}{n^2} < \varepsilon \iff |\frac{1}{n^2} - 0| < \varepsilon \].

Thus $\lim_{n \to \infty} \frac{1}{n^2} = 0$. \(\Box\)

**Remark:** We could have picked $N$ to be any number $\geq \frac{1}{\varepsilon}$, e.g. $N = \frac{2}{\varepsilon^2}, N = \frac{1}{\varepsilon} + 1$, etc.

Ex: Consider the sequence $s_n = (-1)^n$. We expect that this sequence does not converge. Let's prove it.

**Proof:**

Assume, for the sake of contradiction, that $(-1)^n$ converges to $s \in \mathbb{R}$. By defn of convergence, for all $\varepsilon > 0$, there exists $N$ so that $n > N$, $|(-1)^n - s| < \varepsilon$. 


Let $\epsilon = 1$ and choose $N$ so that $n > N$ ensures $|(-1)^n - s| < 1 \iff s - 1 < (-1)^n < s + 1$.

For $n$ even, this implies $1 < s + 1 \Rightarrow 0 < s$. For $n$ odd, this implies $s - 1 < -1 \Rightarrow s < 0$. This is a contradiction. Thus, $(-1)^n$ diverges.

Ex: Consider the sequence $s_n = \frac{2n-1}{3n+2}$. What is the limit?

**Scratchwork:**

$$s_n = \frac{2n-1}{3n+2} = \frac{2 - \frac{1}{n}}{3 + \frac{2}{n}}$$

"these get very small as $n \to \infty$"

$$|s_n - \frac{2}{3}| < \frac{3}{2} \iff \left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| < \frac{3}{2} \iff \left| \frac{6n-3-6n-4}{3(3n+2)} \right| < 3$$

$$\iff \left| \frac{-7}{3(3n+2)} \right| < 3 \iff \frac{7}{3(3n+2)} < 3$$

$$\iff \frac{7}{9n} < 3 \iff \frac{1}{n} < 3 \iff \frac{1}{3} < n$$
\textbf{Proof:}

Fix $\varepsilon > 0$ arbitrary and let $N = \frac{1}{\varepsilon}$. Then, if $n > N$, we have

$$\frac{1}{\varepsilon} < n \implies \frac{7}{3(3n+2)} < \varepsilon \implies |\frac{6n-3-6n-4}{3(3n+2)}| < \varepsilon \implies |S_n - \frac{2}{3}| < \varepsilon.$$

Therefore, $\lim_{n \to \infty} S_n = \frac{2}{3}$. \qed
A special type of sequence is a...

**Def (bounded sequence):** A sequence \( s_n \) is **bounded** if there exists \( M \in \mathbb{R} \) s.t. \( |s_n| \leq M \) for all \( n \).

**Remark:** A sequence is bounded iff the set \( S = \{ s_n : n \in \mathbb{N} \} \) is bounded.

**Thm:** Convergent sequences are bounded.

**Idea of proof:**

\[ s_n \]

\[ s \]

\[ n \in \mathbb{N} \]

\[ S \]

\[ E \]

\[ \exists \]

\[ (-1)^n \]

\[ S = \{-1, 1\} \]

\[ \frac{1}{n} \]

\[ S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \]
Pf:
Suppose $s_n$ is a convergent sequence with limits. By the defn of convergence, for all $\varepsilon > 0$, there exists $N$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.

Let $\varepsilon = \sqrt{2}$. Then there exists $N$ so that $n > N$, $|s_n - s| < \sqrt{2}$.

Since $|s_n| - |s| \leq |s_n - s| \leq |s_n - s| < \sqrt{2}$, so $|s_n| < |s| + \sqrt{2}$ for all $n > N$. $\lfloor N \rfloor = \max \{ n : n \in \mathbb{N} \land n > N \}$.

Define $M = \max \{|s_1|, |s_2|, \ldots, |s_{\lfloor N \rfloor}|, |s| + \sqrt{2}\}$. Then $|s_n| \leq M$ for all $n$, so $s_n$ is a bounded sequence.