Recall:

**Def (convergence):**

- A sequence $s_n$ of real numbers converges to some $s \in \mathbb{R}$ provided that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.

- The number $s$ is the limit of $s_n$, and we write $\lim_{n \to \infty} s_n = s$ or $s_n \to s$.

- A sequence that does not converge to any $s \in \mathbb{R}$ it is said to diverge.
**Def (bounded sequence):** A sequence \( s_n \) is **bounded** if there exists \( M \in \mathbb{R} \) s.t. \( |s_n| \leq M \) for all \( n \).

**Thm:** Convergent sequences are bounded.

Now, we will prove several **limit theorems** that will help us find the limits of more complicated sequences by breaking them into parts.

**Thm (limit of sum is sum of limits):** If \( s_n \) and \( t_n \) are convergent sequences, \( \lim_{n \to \infty} (s_n + t_n) = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n \).

**Ex:** \( \lim_{n \to \infty} \left( \frac{1}{n^2} \frac{\sqrt{2}}{n^2} \right) = \lim_{n \to \infty} \frac{1}{n^2} + \lim_{n \to \infty} \frac{\sqrt{2}}{n^2} = 0 + 0 = 0 \)

**Recall:** triangle inequality \( |a+b| \leq |a|+|b| \).

**Pf:** Let \( s = \lim_{n \to \infty} s_n \) and \( t = \lim_{n \to \infty} t_n \). Fix \( \varepsilon > 0 \). We must show there exists \( N \in \mathbb{R} \) so that \( n > N \) ensures \( |(s_n + t_n) - (s + t)| < \varepsilon \).

Note that \( |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| \).
Since $s_n \to s$ and $t_n \to t$ given $\varepsilon = \frac{\delta}{2} > 0$, there exists $N_s$ and $N_t \in \mathbb{R}$ so that $n > N_s$ ensures $|s_n - s| < \varepsilon$ and $n > N_t$ ensures $|t_n - t| < \varepsilon.$ Let $N = \max\{N_s, N_t\}$. Then for all $n > N,$
\[|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon + \varepsilon = \varepsilon.\quad \Box
\]

Remark: The requirement that $s_n$ and $t_n$ are convergent sequences is necessary. For example, $s_n = (-1)^n$, $t_n = (-1)^{n+1}$.

Then $\lim_{n \to \infty} s_n + t_n = 0$, but $\lim_{n \to \infty} s_n$ and $\lim_{n \to \infty} t_n$ do not exist.
Thm (limit of product is product of limits): If $s_n$ and $t_n$ are convergent sequences, $\lim_{n \to \infty} s_n t_n = \left( \lim_{n \to \infty} s_n \right) \left( \lim_{n \to \infty} t_n \right)$

Exercise

Give an example to show that the assumption that $s_n$ and $t_n$ are convergent sequences is necessary for the previous theorem to be true.

Pf: Let $s = \lim_{n \to \infty} s_n$, $t = \lim_{n \to \infty} t_n$. Fix $\varepsilon > 0$. We must show there exists $N \in \mathbb{N}$ so that $n > N$ ensures $|s_n t_n - s t| < \varepsilon$.

Note that $|s_n t_n - s t| = |s_n t_n - s t_n + s t_n - s t| \\ \leq |s_n t_n - s t_n| + |s t_n - s t| \\ = |s_n| |t_n - t| + |t| |s - s|$

Since $s_n$ is a convergent sequence, it is a bounded sequence, that is there exists $M_s$ so that $|s_n| \leq M_s$ for all $n$. Define $M = \max \{ M_s, |t|, 1 \}$.

$\leftarrow$ ensures $m \geq M_s, m \geq |t|, m > 0$
Combining with estimates above, \(|s_{ntn} - s_t| \leq M|t_{nt} + |M|s_{nt} - s_t|\). For \(\varepsilon = \frac{\varepsilon_n}{2M} > 0\), there exists \(N_s\) and \(N_t\) so that \(n > N_s\) ensures \(|s_n - s| < \varepsilon\) and \(n > N_t\) ensures \(|t_n - t| < \frac{\varepsilon}{M}\). Let \(N = \max \{N_s, N_t\}\). Then for all \(n > N\), \(|s_{ntn} - s_t| \leq M\varepsilon + M\varepsilon = \varepsilon. \) \(\Box\)

**Theorem (limit of quotient is quotient of limits):** If \(s_n\) and \(t_n\) are convergent sequences, \(s_n \neq 0\) for all \(n\), and \(\lim_{n \to \infty} s_n = 0\), then

\[
\lim_{n \to \infty} \left( \frac{t_n}{s_n} \right) = \frac{\lim_{n \to \infty} t_n}{\lim_{n \to \infty} s_n}.
\]

**Proof:** See textbook.
Thm: (basic examples):
(a) \( \lim_{n \to \infty} \left( \frac{1}{n} \right)^p = 0 \) if \( p > 0 \)
(b) \( \lim_{n \to \infty} a^n = 0 \) if \( |a| < 1 \)
(c) \( \lim_{n \to \infty} n^{\frac{1}{n}} = 1 \)
(d) \( \lim_{n \to \infty} a^{\frac{1}{n}} = 1 \) if \( a > 0 \)

Proof: See textbook.

Ex: Find the limit of \( S_n = \frac{n^2 - 2}{n^2 + 2} \) and justify your answer.

Warning: can't immediately use that limit of quotient is quotient of limit.

\[
S_n = \frac{\frac{1}{n} - \frac{2}{n^2}}{1 + \frac{2}{n^2}}
\]

By Thm that limit of quotient is quotient of limits, we have \( S_n \to 0 \).
Ex: What is the limit of \( s_n = n^2 \)?

**Def (diverges to \( \pm \infty \) or \( -\infty \)):** A sequence \( s_n \) diverges to \( \pm \infty \) if for all \( M > 0 \) there exists \( N \in \mathbb{R} \) so that \( n > N \) ensures \( s_n > M \). We write \( \lim_{n \to \infty} s_n = \pm \infty \).

Likewise, a sequence \( s_n \) diverges to \( -\infty \) if for all \( M < 0 \) there exists \( N \) so that \( n > N \) ensures \( s_n < M \).

We write \( \lim_{n \to \infty} s_n = -\infty \).

**Remark:**

- If \( s_n \) diverges to \( \pm \infty \), it does not converge.
- We will say that \( s_n \) "has a limit" or "the limit of \( s_n \) exists" if either
  1. \( s_n \) converges
  2. \( s_n \) diverges to \( \pm \infty \)

\[
\lim_{n \to \infty} s_n \in \mathbb{R} \\
\lim_{n \to \infty} s_n \in \{\pm \infty, -\infty\}
\]

A few limit theorems for sequences that diverge to \( \pm \infty \) or \( -\infty \):

- **Thm:** Suppose \( \lim_{n \to \infty} s_n = \pm \infty \) and \( \lim_{n \to \infty} s_n > 0 \). Then, \( \lim_{n \to \infty} s_n t_n = \pm \infty \).
- **Pf:** HW.
**Thm:** Suppose $s_n$ is a sequence of positive real numbers. Then $\lim_{n \to \infty} s_n = +\infty$ if $\lim_{n \to \infty} \frac{1}{s_n} = 0$.

**Pf:** First, suppose $\lim_{n \to \infty} s_n = +\infty$. Fix $\epsilon > 0$. Note that $|\frac{1}{s_n} - 0| < \epsilon \iff \frac{1}{s_n} < \epsilon \iff \frac{1}{\epsilon} < s_n$. Since $s_n$ diverges to $+\infty$, there exists $N$ s.t. $n > N$ ensures $s_n > \frac{1}{\epsilon} \iff |\frac{1}{s_n} - 0| < \epsilon$. Thus, $\lim_{n \to \infty} \frac{1}{s_n} = 0$.

Next, suppose $\lim_{n \to \infty} \frac{1}{s_n} = 0$. Fix $M > 0$. Note that $s_n > M \iff \frac{1}{s_n} < \frac{1}{M} \iff |\frac{1}{s_n} - 0| < \frac{1}{M}$. Since $\frac{1}{s_n}$ converges to 0, there exists $N$ s.t. $n > N$ ensure $d |\frac{1}{s_n} - 0| < \frac{1}{M} \iff s_n > M$. Thus, $\lim_{n \to \infty} s_n = +\infty$.

**Thm:** If $\lim_{n \to \infty} s_n = +\infty$, then $\lim_{n \to \infty} (-s_n) = -\infty$.

**Pf:** Fix $M < 0$. Note that $(-s_n) < M \iff s_n > -M$. Since $s_n$ diverges to $+\infty$, there exists $N$ s.t. $n > N$, $s_n > -M \iff (-s_n) < M$. Thus $\lim_{n \to \infty} -s_n = -\infty$.

Next time: monotone and Cauchy sequences.