Midterm 1 Solutions  
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1. 7 points  
A nonempty set \( S \subseteq \mathbb{R} \) is bounded if it is bounded above and bounded below.

2. 8 points  
By defn of \( A + B \) and the supremum, \( \sup(A+B) \) is an upper bound for \( A + B \), so \( a + b \leq \sup(A+B) \) \( \iff \) \( a \leq \sup(A+B) - b \) for all \( a \in A, b \in B \). Thus, for all \( b \in B \), \( \sup(A+B) - b \) is an upper bound for \( A \).

3. 10 points  
By defn of \( \sup(A) \) as the least upper bound for \( A \), \( \sup(A) \leq \sup(A+B) - b \) for all \( b \in B \). Thus  
\[ b \leq \sup(A+B) - \sup(A) \]  
for all \( b \in B \).

This shows \( \sup(A+B) - \sup(A) \) is an upper bound for \( B \), hence \( \sup(B) \leq \sup(A+B) - \sup(A) \).
This shows \( \sup(A) + \sup(B) \leq \sup(A + B) \).

To see the other inequality, note that, since \( \sup(A) \geq a \ \forall \ a \in A \) and \( \sup(B) \geq b \ \forall \ b \in B \),

\[
\sup(A) + \sup(B) \geq a + b \ \forall \ a \in A, b \in B.
\]

Thus \( \sup(A) + \sup(B) \) is an upper bound for \( A + B \), hence

\[
\sup(A) + \sup(B) \leq \sup(A + B).
\]

\[2\] 5 points

\( \boxed{\text{2.} \text{ A sequence } s_n \text{ is convergent if there exists } s \in \mathbb{R}\text{ s.t. } \forall \epsilon > 0, \exists N \in \mathbb{N}\text{ s.t. } n \geq N \text{ ensures } |s_n - s| < \epsilon.} \]

\[6\] 7 points

\( \boxed{\text{6.} \text{ Fix } \epsilon > 0. \text{ Note that }\]
\[
\left| \frac{n-1}{n^2-1} - 0 \right| = \left| \frac{1}{n+1} - 0 \right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon
\]

if \( n > \frac{1}{\epsilon} \). Thus, for \( N = \frac{1}{\epsilon} \), \( n \geq N \) ensures that \( \left| \frac{n-1}{n^2-1} - 0 \right| < \epsilon \).

Since \( \epsilon > 0 \) was arbitrary, this shows \( \lim_{n \to \infty} \frac{n-1}{n^2-1} = 0 \).
18 points

(2) Assume, for the sake of contradiction, that \( x_n \) converges to \( x \in \mathbb{R} \). Then for \( \varepsilon = 1 \), there exists \( N \in \mathbb{R} \) s.t.

\[ |x_n - x| < 3 \quad \forall \ n > N. \]

In particular, \( x_n < x + 3 = x + 1 \ \forall \ n > N \).
However, \( x_n = (n-1)^2 + 1 \geq (n-1)^2 \geq n - 1 \)
for all \( n \in \mathbb{N} \). Thus, for
\( n > \max\{x + 2, N\} \), \( x_n \geq (x + 2) - 1 = x + 1 \).
Since it is impossible to have both \( x_n < x + 1 \) and \( x_n > x + 1 \), we have found a contradiction. Thus \( x_n \) does not converge to \( x \). Since \( x \) was arbitrary, this shows \( x_n \) does not converge.

10 points

(3) Fix \( \varepsilon > 0 \). Note that, by the reverse triangle inequality,

\[ |t| - |t_n| \leq |t - t_n|. \]
Since \( \lim_{n \to \infty} t_n = t \), \( \exists \ \mathbb{N} \in \mathbb{R} \text{ s.t. } n > N \) ensures \( |t - t_n| < \varepsilon \), hence \( |t| - |t_n| < \varepsilon \).
Since \( \varepsilon > 0 \) was arbitrary, this shows \( \lim |t_n| = |t| \).

15 points

(b) The converse is not true.
Let \( t_n = (-1)^n \). Then \( |t_n| = 1 \) is a convergent sequence, but \( t_n \) is not a convergent sequence.

4 points

(a) Assume for the sake of contradiction that \( \lim_{n \to \infty} s_n = s < a \). Then if we let \( \varepsilon = a - s > 0 \), there exists \( N \) so that \( n > N \) ensures \( |s_n - s| < \varepsilon \) \( \iff \) \( s - \varepsilon < s_n < s + \varepsilon \).
Thus \( \exists n \in \mathbb{N} : s_n \geq a \) \( \subseteq \{1, 2, 3, \ldots, \infty \} \).
Hence, it is a finite set, so \( \exists n \in \mathbb{N} : s_n < a \)
must be infinite. This is a contradiction.

(b) If \( s_n \) and \( t_n \) are convergent sequences, then

5 points \( \lim (s_n + t_n) = (\lim s_n) + (\lim t_n) \).
Define \( r_n := t_n - s_n \). Then \( r_n \geq 0 \) for all but finitely many \( n \in \mathbb{N} \). Furthermore, since the limit of the product is the product of the limits, \( -s_n \) is a convergent sequence, and, by part \( b \), \( r_n = t_n - s_n \) is convergent and \( \lim r_n = \lim t_n - \lim s_n \).

Thus, by part \( a \),
\[
0 \leq \lim r_n = \lim t_n - \lim s_n.
\]
Hence \( \lim s_n \leq \lim t_n \).

\( \text{5 points} \)

\( a \) (i) \( \max(C) \) does not exist
\[
\sup(C) = +\infty
\]
\( 15 \text{ points} \)
(ii) \( \min(C) = \inf(C) = 0 \).
\( 15 \text{ points} \)

\( b \) (i) False. Consider \( s_n = \frac{(-1)^n}{n} \).
(ii) False. Consider \( s_n = (-1)^n \).
\( 5 \text{ points} \)