Midterm 2 Solutions
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1. For all \( n \in \mathbb{N} \),
   \[
   \cos\left(\frac{n\pi}{2}\right) = \begin{cases} 
   0 & \text{if } n \text{ is odd} \\
   1 & \text{if } n = 4k \text{ for } k \in \mathbb{N} \\
   -1 & \text{if } n = 4k+2 \text{ for } k \in \mathbb{N}.
   \end{cases}
   \]

(a) Since the constant sequences
   \((0,0,0,0,...), (1,1,1,...), \text{ and } (-1,-1,-1,-1,...)\)

are subsequences of \(0,0,1,-1\), and \(0,0,1,-1\)

are subsequential limits.

It remains to show that there are no other subsequential limits.

Fix \( \xi \in \mathbb{R}\setminus\{1,0,1,3\} \).

Let \( \varepsilon = \min \{ |\xi - (-1)|, |\xi - 0|, |\xi - 1|, |\xi - 3| \} > 0 \).
Then since \( |\xi - (-1)| \geq \varepsilon \), \( |\xi - 0| \geq \varepsilon \), and
\( |\xi - 1| \geq \varepsilon \), we have
\[
\{ n : |\xi n - \xi| < \varepsilon \} = \emptyset.
\]
In particular, this set is finite. Thus, by the main subsequences
theorem, \( s \) is not a subsequential limit.

\[ \begin{align*}
\text{b)} & \quad \text{Since } \liminf_{n \to \infty} a_n \text{ and } \limsup_{n \to \infty} a_n \text{ are the smallest and largest subsequential limits, they are } -1 \text{ and } 1, \text{ respectively.} \\
\text{c)} & \quad \text{Since the set of subsequential limits contains more than one element, the limit does not exist.}
\end{align*} \]

\[ \begin{align*}
\text{2)} & \quad \text{Suppose } a \in A, \text{ so } s_n \geq a \text{ for all but finitely many } n. \text{ Then any convergent subsequential } s_{n_k} \text{ also satisfies } s_{n_k} \geq a \text{ for all but finitely many } k. \text{ Hence } \lim_{k \to \infty} s_{n_k} \geq a. \text{ Thus all subsequential limits are } \geq a. \\
& \quad \text{Since } \lim_{n \to \infty} s_n \text{ is a subsequential limit, } \lim_{n \to \infty} s_n \geq a. \text{ Since } a \text{ was arbitrary, this shows } \lim_{n \to \infty} s_n \text{ is an upper bound for } A. \text{ Since } \sup(A) \text{ is the least upper bound, } \sup(A) \leq \lim_{n \to \infty} s_n.
\end{align*} \]
We prove the contrapositive. If \( r \not\in A \), then there are infinitely many elements of the sequence so that \( s_n < r \). In particular, there exists a subsequence \( s_{n_k} \) of \( s_n \) so that \( s_{n_k} < r \) for all \( k \). Hence, \( \lim_{k \to \infty} s_{n_k} = r \). Since \( \lim_{n \to \infty} s_n \) is the smallest subsequential limit, \( \lim_{n \to \infty} s_n \leq r \).

By part (a), \( \lim_{n \to \infty} s_n \) is an upper bound for \( A \), so it remains to show it is the least upper bound. Suppose there exists \( m < \lim_{n \to \infty} s_n \) so that \( m \) is an upper bound for \( A \). By denseness of \( \mathbb{Q} \) in \( \mathbb{R} \), there exists \( r \in \mathbb{Q} \) s.t. \( m < r < \lim_{n \to \infty} s_n \). By part (b), \( r \in A \). This contradicts the fact that \( m \) was an upper bound for \( A \). Therefore, \( \lim_{n \to \infty} s_n \) must be the least upper bound for \( A \).

See HW3, Q14
A series $\sum_{k=1}^{\infty} a_k$ converges if the partial sum sequence $s_n = \sum_{k=1}^{n} a_k$ converges.

Since $|b_k| \geq 0$ for all $k$, $s_n = \sum_{k=1}^{n} |b_k| \leq \sum_{k=1}^{n} b_k = s_n - 1$. Thus $s_n$ is increasing, so $\lim_{n \to \infty} s_n$ is either a real number or $+\infty$. Consequently, $\sum_{k=1}^{\infty} |b_k| l$ is either convergent and diverges to $+\infty$.

Suppose $\sum_{k=1}^{\infty} |b_k| l$ is convergent. By the Cauchy Criterion, $\forall \varepsilon > 0$, $\exists N$ s.t. $n > m > N$ implies $\sum_{k=m+1}^{n} |a_k| \leq \varepsilon$.

Since $\sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} |a_k| = \sum_{k=m+1}^{n} |a_k|$, this ensures that $\forall \varepsilon > 0$, $\exists N$ s.t. $n > m > N$ implies $\sum_{k=m+1}^{n} |a_k| \leq \varepsilon$.

Applying the Cauchy Criterion again, we obtain that $\sum_{k=1}^{\infty} a_k$ is convergent.
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6.1 \((\sqrt{2}, \sqrt{4}, \sqrt{6}, ... , \sqrt{2k}, ... )\)
\(= (\frac{\sqrt{2}}{2}, \frac{\sqrt{4}}{4}, \frac{\sqrt{6}}{6}, ... , \frac{\sqrt{2k}}{2k}, ... )\)

(ii) \(\varepsilon - 1, 1\)

(iii) 1

(iv) -1

(b) True

(ii) False: consider the sequence \( (0, 1, 1, 0, 2, 0, 3, ... ) \)