Craig Office Hours, 5/27/22

26 ii)

\[ f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \]

Since a function is continuous if it is continuous at every point in its domain, to show a function is discontinuous it suffices to show \( \exists x_0 \in \text{dom}(f) \) s.t. \( f \) is discontinuous at \( x_0 \).

We will show \( f \) is discontinuous at \( x_0 = 0 \).

Def: \( f \) is discontinuous at \( x_0 \in \text{dom}(f) \) if \( \exists \varepsilon > 0 \) s.t. \( \forall \delta > 0, \exists x \in \text{dom}(f) \) with \( |x - x_0| < \delta \) and \( |f(x) - f(x_0)| \geq \varepsilon \).
Let \( \varepsilon = 1 \). Fix \( \delta > 0 \) arbitrary. Let \( x = \min\{\frac{\varepsilon}{2}, 1\} \). Then \( |x - x_0| = x < \delta \), and \( f(x) = \max \{\varepsilon \frac{2}{8}, 1\} \delta \). Then,

\[
|f(x) - f(x_0)| = f(x) \geq 1 = \varepsilon .
\]

Since \( \delta > 0 \) was arbitrary, this shows that for all \( \delta > 0 \), \( \exists x \in \text{dom}(f) \) s.t. \( |x - x_0| < \delta \) and \( |f(x) - f(x_0)| \geq \varepsilon \).
This shows $f$ is discontinuous at $x_0$.

Now, consider

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

**Proof.** We will show $f$ is discontinuous at $x_0 = 0$. Fix $\varepsilon = 2$. Fix $\delta > 0$ arbitrary. Let $x = \frac{-\delta}{2}$. Then $|x - x_0| = \frac{\delta}{2} < \delta$, and $|f(x) - f(x_0)| = |(-1) - 1| = 2 = 2 \geq \varepsilon$.

Since $\delta > 0$ was arbitrary, this shows that, for all $x_0 > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon$. Thus $f$ is discontinuous.
at $x_0$.

1. $S = \{ \sqrt{5}^r : r \in \mathbb{Q} \}$

2. WTS: \( \forall a, b \in \mathbb{R} \) with \( a < b \), \( \exists s \in S \) s.t. \( a < s < b \).

_Pf:_ Since \( a < b \), we also have \( \frac{a}{\sqrt{5}} < \frac{b}{\sqrt{5}} \). Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), \( \exists r \in \mathbb{R} \) s.t. \( \frac{a}{\sqrt{5}} < r < \frac{b}{\sqrt{5}} \). Thus, \( a < \sqrt{5}r < b \). This shows \( \exists s \in S \) s.t. \( a < s < b \).

**Scratchwork:**

Want: \( a < s < b \) for \( s \in S \).

\[
\begin{align*}
& a < s < b \quad \text{for } s \in S. \\
\implies & \quad a < \sqrt{5}r < b \quad \text{for } r \in \mathbb{R}.
\end{align*}
\]

\[
\begin{align*}
& \frac{a}{\sqrt{5}} < r < \frac{b}{\sqrt{5}} \quad \text{for } r \in \mathbb{R}.
\end{align*}
\]

\( f(x) = \begin{cases} 
0 & \text{if } x \in S^c \\
1 & \text{if } x \in S
\end{cases} \)
WTS: For all $x_0 \in S^c$, $f$ is discontinuous at $x_0$.

**Proof:** Fix $x_0 \in S^c$. Fix $\varepsilon = 1$. Fix $\delta > 0$.
Note that $|x-x_0| < \delta \iff x-\delta < x < x_0 + \delta$.

By part (a), for $a = x_0 - \delta$ and $b = x_0 + \delta$, 
$\exists x \in S$ s.t. $a < x < b$, so $\exists x \in S$ s.t. 
$|x-x_0| < \delta$. Then $|f(x) - f(x_0)| = 1 = \varepsilon$.

Since $\delta > 0$ was arbitrary, this shows that, 
for all $\delta > 0$, $\exists x \in \text{dom}(f)$ s.t. $|x-x_0| < \delta$ 
and $|f(x) - f(x_0)| \geq \varepsilon$. Thus $f$ is discontinuous 
at $x_0$. 
(14)

\( \circ \) WTS \( \forall \varepsilon > 0, a \in \mathbb{R}, \{r \in \mathbb{Q} : |r - a| < \varepsilon\} \) contains infinitely many elements.

\( \text{Proof:} \) Fix \( \varepsilon > 0 \) and \( a \in \mathbb{R} \). Note that \( \frac{|r - a|}{\varepsilon} < 1 \iff a - \varepsilon < r < a + \varepsilon \).

By density of \( \mathbb{Q} \) in \( \mathbb{R} \), \( \exists r_2 \in \mathbb{Q} \) s.t. \( a < r_2 < a + \varepsilon \). In particular, \( |r_2 - a| < \varepsilon \).

Now, by density of \( \mathbb{Q} \) in \( \mathbb{R} \), \( \exists r_2 \in \mathbb{Q} \) s.t. \( a < r_2 < r_1 \). In particular, \( |r_2 - a| < \varepsilon \) and \( r_2 \neq r_1 \).