Question 1

Let $s_n$ be the sequence defined in the following figure from the textbook:

(a) Find the set $S$ of subsequential limits of $s_n$. Justify your answer.
(b) Determine $\limsup s_n$ and $\liminf s_n$. Justify your answer.

Question 2

Suppose $s_n$ is a bounded sequence and define $s = \sup\{s_n : n \in \mathbb{N}\}$.

(a) State the definition of a bounded sequence.
(b) Suppose $s_n < s$ for all $n \in \mathbb{N}$. Prove that, for all $k \in \mathbb{N}$, the set $B_k := \{s_n : s_n > s - \frac{1}{k}\}$ has infinitely many elements.
   (Hint: Prove the result by contradiction, assuming that for some $k \in \mathbb{N}$, $B_k$ has finitely many elements. Consider the cases where $B_k$ has zero elements and $B_k$ has a positive number of elements separately.)
(c) Suppose $s_n < s$ for all $n \in \mathbb{N}$. Use part (b) to show that there exists a subsequence of $s_n$ converging to $s$.
(d) Now suppose $s_n = s$ for some $n \in \mathbb{N}$. Give an example of such a sequence that doesn’t have a subsequence converging to $s$.

Question 3

Recall that a sequence $s_n$ is a Cauchy sequence if and only if

\[ \text{for all } \epsilon > 0 \text{ there exists } N \in \mathbb{R} \text{ so that } n > m > N \text{ ensures } |s_n - s_m| < \epsilon. \]

(In our definition from class, we did not state $n > m > N$, but instead $n, m > N$, but we may assume $n > m > N$ without loss of generality.)
(a) State the Cauchy Criterion theorem.

(b) Use the above definition of a Cauchy sequence to prove the Cauchy Criterion.

(c) Use the Cauchy Criterion to prove the following corollary:

**COROLLARY 1.** If the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k \to +\infty} a_k = 0$.

**Question 4 (Quiz 3 Q3)**

(a) Suppose $s_n$ has a subsequence $s_{n_k}$ that is bounded. Show that this implies $s_n$ has a convergent subsequence.

(b) Suppose that $s_n$ has no convergent subsequences. Prove that $\lim_{n \to +\infty} |s_n| = +\infty$.

(Hint: prove the result by contradiction, by showing that if $\lim_{n \to +\infty} |s_n| \neq +\infty$, then $s_n$ has a bounded subsequence.)

**Question 5**

**Lightning Round!**
You do not need to show your work or justify your answers.

(a) Suppose $a_n = (\sin \frac{\pi n}{2})^2$ and $b_n = (-1)^n$.

(i) What is the set of subsequential limits of $a_n$? $b_n$?

(ii) Does $a_n + b_n$ converge? Does $2a_n + b_n$ converge?

(b) Determine if the following statements are true or false. If they are false, provide a counterexample.

(i) If $s_n$ is a monotone increasing sequence, then $\lim s_n = \inf \{s_n : n \in \mathbb{N}\}$.

(ii) Suppose $s_n$ is not a bounded sequence. Then either $+\infty$ or $-\infty$ is a subsequential limit.

Other useful questions:

**Question 6**

Consider the sequences defined as follows:

$$a_n = (-1)^{n+1}, \quad b_n = \frac{-1}{n}, \quad c_n = 2n, \quad d_n = \frac{3n+1}{4n-1}.$$ 

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, give its set of subsequential limits. Justify your answer.

(c) For each sequence, give its $\lim \inf$ and $\lim \sup$. Justify your answer.

(d) Which of the sequences converges? Diverges to $+\infty$? Diverges to $-\infty$? Justify your answer.

(e) Which of the sequences is bounded? Justify your answer.
Question 7

On previous homework assignments, you proved the following results:

\[ \lim_{n \to +\infty} r^n = \begin{cases} 
0 & \text{if } |r| < 1 \\
1 & \text{if } |r| = 1 \\
+\infty & \text{if } r > 1 \\
does not exist & \text{if } r \leq -1 
\] 

and

\[ \text{for } r \neq 1, \sum_{k=1}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}. \]

(a) Prove that for \( |r| < 1 \), \( \sum_{k=1}^{\infty} r^k = \frac{1}{1-r} \).

(b) Prove that for \( |r| > 1 \), \( \sum_{k=1}^{\infty} r^k \) does not converge. (Hint: Show that \( \lim_{k \to +\infty} r^k \neq 0 \) and use the corollary from Q3(c).)

Question 4

Suppose \( \sum_{k=1}^{\infty} a_k = A \) and \( \sum_{k=1}^{\infty} b_k = B \) for \( A, B \in \mathbb{R} \).

(a) Use the limit theorems for sequences to prove that \( \sum_{k=1}^{\infty} (a_k + b_k) = A + B \).

(b) Use the limit theorems for sequences to prove that for \( c \in \mathbb{R}, \sum_{k=1}^{\infty} ca_k = cA \).