(a) **Claim:** \( S = \{0\} \cup \{ \frac{1}{x} : x \in \mathbb{N} \} \)

Since \( S_{1/k} = \frac{1}{k} \) is a subsequence, \( 0 \in S \). Since \( S_{1/x} = \frac{1}{x} \) is a subsequence for all \( x \in \mathbb{N} \) and \( \frac{1}{x} \in S \).

It remains to show no other real number \( \pm \infty \) belongs to \( S \).

Neither \( \infty \) nor \( -\infty \) belong to \( S \), since the sequence is bounded.

Suppose \( a \in S \) for some \( a \in \mathbb{R} \). By the main subsequence theorem, it suffices to show \( \exists \varepsilon_0 > 0 \) so that \( |a - s_n| > \varepsilon_0 \) for all \( n \).

- If \( a > 1 \), then \( |a - s_n| > |a - 1| = \varepsilon_0 \).
- If \( a < 0 \), then \( |a - s_n| > |a| = \varepsilon_0 \).
- If \( \frac{1}{x} > a > \frac{1}{x+1} \) for some \( x \in \mathbb{N} \), then
This completes the proof.

(6) \( \limsup_n s_n = \max(s) = 1 \)

\( \liminf_n s_n = \min(s) = 0 \)

(6)(a) \( s_n \) is a bounded sequence if \( \exists M > 0 \) s.t. \( \|s_n\| < M \) \( \forall n \in \mathbb{N} \).

(6) Assume for the sake of contradiction that \( \exists k \in \mathbb{N} \) s.t. \( B_k \) is such that \( s_n \geq s - \frac{12}{k^3} \) has finitely many elements.

Case 1: \( B_k \) has zero elements.
Then \( s_n \leq s - \frac{18}{k^3} \) for all \( n \in \mathbb{N} \).
This contradicts the fact that \( s \) is the least upper bound.

Case 2: \( B_k \) has a nonzero number of elements. Then \( B_k \) has a maximum \( M_k = \max B_k \). Since \( s_n < s \) for all \( n \), \( M_k < s \). Also, note that if \( s_n \in B_k \), then...
Thus, $M_k$ is an upper bound for $\{s_n: n \in \mathbb{N}\}$. Since $M_k < s$, this contradicts that $s$ was the least upper bound.

Therefore, $B_k$ has infinitely many elements for all $k \in \mathbb{N}$.

Since $B_1$ has infinitely many elements, we may choose $s_{n_1}$ so $s - 1 < s_{n_1} < s$.

Since $B_2$ has infinitely many elements, we may choose $s_{n_2}$ so $n_2 > n_1$ and $s - \frac{1}{2} < s_{n_2} < s$.

Since $B_k$ has infinitely many elements, we may choose $s_{n_k}$ so $n_k > n_{k-1}$ and $s - \frac{1}{k} < s_{n_k} < s$.

By construction, $s_{n_k}$ is a subsequence of $s_n$ and, for all $k \in \mathbb{N}$, $s - \frac{1}{k} < s_{n_k} < s$. Thus, by the squeeze lemma, $\lim_{k \to \infty} s_{n_k} = s$. Therefore, there exists a subsequence of $s_n$ converging to $s$. 
Define \( s_n = \frac{1}{n} \). Then \( s = \sup \{ s_n : n \in \mathbb{N} \} = 1 \), but since \( \lim_{n \to \infty} s_n = 0 \), all subsequences of \( s_n \) converge to 0.

See Lecture 11.

\[ b \quad \sum_{k=1}^{\infty} a_k \text{ is convergent} \]
\[ \iff \quad \sum_{n=1}^{\infty} s_n \text{ converges} \]
\[ \iff \quad s_n \text{ is Cauchy} \]
\[ \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ so that } n > M > N \]
ensures \( |s_n - s_M| < \varepsilon \)
\[ \downarrow \]
\[ \sum_{k=1}^{n} a_k - \sum_{k=1}^{N} a_k = \sum_{k=M+1}^{n} a_k \]
\[ \forall \varepsilon > 0, \exists N \in \mathbb{R} \text{ so that } n > M > N \]
ensures \( |\sum_{k=M}^{n} a_k| < \varepsilon \)
(c) Suppose \( \lim_{k \to \infty} a_k \) is convergent. WTS \( \lim_{k \to \infty} a_k = 0 \). Fix \( \varepsilon > 0 \). By part (b), \( \exists N \) s.t. \( n > m > N \) implies \( \sum_{k=m+1}^{n} |a_k| < \varepsilon \). In particular, \( \exists N \) s.t. \( m > N \) and \( n = m + 1 \) implies \( \sum_{k=m+1}^{n} |a_k| < \varepsilon \), so \( \sum_{k=m+1}^{n} |a_k| < \varepsilon \). Thus \( \lim_{k \to \infty} a_k = 0 \).

(d) If \( S_{n_k} \) is bounded, by Bolzano–Weierstrass, \( S_{n_k} \) must have a convergent subsequence \( S_{n_{k,E}} \). Since \( S_{n_{k,E}} \) is also a subsequence of \( S_n \), \( S_n \) has a convergent subsequence.

(b) Suppose \( |s_n| \) does not diverge to \( +\infty \). Then \( \exists M > 0 \) s.t. \( \forall N, \exists n > N \) for which \( |s_n| \leq M \). Since \( |s_n| \leq M \) for all \( n \in \mathbb{N} \), this implies there exist infinitely many \( n \in \mathbb{N} \) for which \( 0 \leq |s_n| \leq M \). Consequently, there exists a subsequence \( S_{n_k} \) for which \( 0 \leq |s_{n_k}| \leq M \) \( \forall k \in \mathbb{N} \). Therefore \( S_{n_k} \) is a bounded sequence, so by part (d), \( S_n \) must have a convergent subsequence.
5)

a)
i) The subsequential limits of \( a_n, b_n \) are \([-1,1], [-1,1] \).

\[ a_n + b_n = \begin{cases} 0 & \text{odd} \\ 2 & \text{even} \end{cases} \]
\[ 2a_n + b_n = \begin{cases} 1 & \text{odd} \\ 1 & \text{even} \end{cases} \]

Thus \( a_n + b_n \) doesn't converge but \( 2a_n + b_n \) does converge.

b)

i) False. Consider \( s_n = \frac{-1}{n} \). Then \( \lim s_n = 0 \neq -1 = \inf \{ s_n : n \in \mathbb{N} \} \).

ii) True. See main subsequences theorem.

6)
(a) | Sequence | Monotone subsequence |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>$(1, 1, 1, \ldots)$</td>
</tr>
<tr>
<td>$b_n$</td>
<td>$-\frac{1}{n}$</td>
</tr>
<tr>
<td>$c_n$</td>
<td>$2n$</td>
</tr>
<tr>
<td>$d_n$</td>
<td>$\frac{3n+1}{4n-1} = \frac{3 + \frac{1}{n}}{4 - \frac{1}{n}}$</td>
</tr>
</tbody>
</table>

(b) | Sequence | Set | Justification |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>$E = {1, 1, 1, \ldots}$</td>
<td>$1$ and $-1$ are clearly subsequential limits, since the constant sequences $(1, 1, 1, \ldots)$ and $(-1, -1, -1, \ldots)$ are subsequences of $a_n$. For $\varepsilon = \frac{1}{2}$, ${n :</td>
</tr>
<tr>
<td>$b_n$</td>
<td>$\mathbb{Q}^3$</td>
<td>If a sequence had a limit, then all subsequences have the same limit.</td>
</tr>
<tr>
<td>$c_n$</td>
<td>$\mathbb{R}^2$</td>
<td></td>
</tr>
<tr>
<td>$d_n$</td>
<td>$\mathbb{Q}^3$</td>
<td></td>
</tr>
</tbody>
</table>

(c) \[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} \sup \{a_n : n > N_3\} = \lim_{n \to \infty} 1 = 1 \\
\liminf_{n \to \infty} a_n = \lim_{n \to \infty} \inf \{a_n : n > N_3\} = \lim_{n \to \infty} -1 = -1
\]
Since the limits of $b_n$, $c_n$, and $d_n$ exist, their limsup's and liminf's must coincide with their limits. Thus,

\[
\begin{align*}
\limsup_{n \to \infty} b_n &= \lim_{n \to \infty} b_n = 0 \\
\limsup_{n \to \infty} c_n &= \lim_{n \to \infty} c_n = +\infty \\
\limsup_{n \to \infty} d_n &= \lim_{n \to \infty} d_n = \frac{3}{4}
\end{align*}
\]

\(\square\) \textit{an} does not converge, since its set of subsequential limits contains more than one element.

\(b_n\) converges to 0, \(d_n\) converges to \(\frac{3}{4}\), \(c_n\) diverged to \(+\infty\).

\(\square\) \text{\textit{lan}} \leq 1 \ \forall n \in \mathbb{N}, \text{so it is bounded.} \\
\text{\textit{bn} and \textit{cn} are convergent, hence bounded.}

\(c_n\) is not bounded, since it diverges to \(+\infty\).
7) Define $s_n = \sum_{k=1}^{n} r^k$.

- $\sum_{k=1}^{\infty} r^k = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1 - 0}{1 - r} = \frac{1}{1 - r}$.

8) By the corollary, if $\sum_{k=1}^{\infty} r^k$ converges, then $\lim_{k \to \infty} r^k = 0$. Thus, if we can show $\lim_{k \to \infty} r^k \neq 0$, we must have $\sum_{k=1}^{\infty} r^k$ doesn't converge. If $r > 1$, $\lim_{k \to \infty} r^k = +\infty$ and if $r < -1$ $\lim_{k \to \infty} r^k$ does not exist. Thus, if $|r| > 1$, $\lim_{k \to \infty} r^k \neq 0$.

8) Define $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$.

- $\sum_{k=1}^{\infty} (a_k + b_k) = \lim_{n \to \infty} (S_n + T_n) = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n = A + B$

- $\sum_{k=1}^{\infty} c a_k = \lim_{n \to \infty} (c S_n) = c \lim_{n \to \infty} s_n = cA$. 

- $\sum_{k=1}^{\infty} c b_k = \lim_{n \to \infty} (c T_n) = c \lim_{n \to \infty} t_n = cB$. 

- $\sum_{k=1}^{\infty} (a_k + b_k) = \lim_{n \to \infty} (S_n + T_n) = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n = A + B$.