Math 119a: Homework 1
Due Tuesday, October 2nd

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam.

Question 1* (Similar to 1.2)

For each of the following differential equations, find all equilibrium solutions and determine whether they are sinks, sources, or neither. Then sketch the phase portrait. (Make sure you can do this without a calculator, since no calculators will be allowed during the exams.)

(a) \( x' = 1 - x^{14} \)
(b) \( x' = e^{-x} \sin(x) \)
(c) \( x' = 1 - 2 \cos(x) \)

Question 2 (Similar to 1.9)

Consider a first-order linear equation of the form \( x' = ax + f(t) \), where \( a \in \mathbb{R} \) and \( f(t) \) is a continuous function. Suppose \( y(t) \) is a solution of this equation. Prove that the general solution is \( y(t) + ke^{at} \) for some \( k \in \mathbb{R} \) is arbitrary.

Question 3* (Similar to Strogatz, 2.3.4)

For certain species of organisms, the effective growth rate \( x'/x \) is highest at intermediate \( x \), and at this value of \( x \), the growth rate is positive. This is called the Allee effect (Edelstein-Keshet 1988). For example, imagine that it is too hard to find mates when \( x \) is very small, and there is too much competition for food and other resources when \( x \) is large.

(a) Give conditions on the parameters \( r, a, \) and \( b \) so that \( x'/x = r - a(x - b)^{2} \) provides an example of the Allee effect.

(b) Find all equilibrium solutions of the system and determine whether they are sinks, sources, or neither.

(c) Sketch a few solutions \( x(t) \) for different initial conditions. In your sketch, include all equilibrium solutions, as well as examples of solutions that tend toward sinks and tend away from sources. You do not need to include explicit formulas for the curves you sketch. Consider the cases \( b \leq \sqrt{r/a} \) and \( b > \sqrt{r/a} \) separately.

(d) Compare the solutions \( x(t) \) to those found for the logistic equation. What is the main difference?