Recall:

**Def:** Two linear systems \( X' = AX \) and \( Y' = BY \), \( A, B \in \text{Mn}(\mathbf{R}) \), are equivalent up to coordinate change if \( \exists \ T \in \text{Mn}(\mathbf{R}) \) invertible so that \( B = T^{-1}AT \).

In this case, \( Y(t) \) solves \( Y' = BY \Rightarrow X(t) = TY(t) \) solves \( X' = AX \)
\( X(t) \) solves \( X' = AX \Rightarrow Y(t) = T^{-1}X(t) \) solves \( Y' = BY \)

These solutions are equivalent, up to rotations/ reflections/scalings.

**Thm:** Given \( A \in \text{Mn}_2(\mathbf{R}) \), \( X' = AX \) is equivalent up to coordinate change to one of the following systems:

1. real, distinct eigenvalues \( \lambda_1, \lambda_2 \): \( Y' = (\lambda_1, 0)Y \\
   (0, \lambda_2) \)
2. complex eigenvalues \( \alpha \pm i\beta \): \( Y' = (\alpha, \beta)Y \\
   (-\beta, \alpha) \)
repeated real eigenvalues

(i) 2 linearly independent eigenvectors
\[ Y' = (\lambda A) Y \]
\[ Y' = (\lambda^2 A) Y \]

(ii) 1 eigenvector

These are known as the canonical forms of planar linear systems.

Higher dimensional systems, \( X' = AX, A \in \mathbb{M}_n(\mathbb{R}) \)

**Definition (Matrix Exponential):** Given \( A \in \mathbb{M}_n(\mathbb{R}) \),
\[ e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad e \in \mathbb{M}_n(\mathbb{R}) \]

**Lemma:**

(a) If \( AB = BA \), then \( e^{A+B} = e^A e^B \)

(b) For all \( A \in \mathbb{M}_n(\mathbb{R}) \), \( e^A \) is invertible and \( (e^A)^{-1} = e^{-A} \).

**Proposition:** \( \frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA} A \)

**Major Result #5**

**Theorem (Fundamental Theorem for Linear Systems):**
Given \( A \in \mathbb{M}_n(\mathbb{R}) \), the general solution of \( X' = AX \) is
\[ X(t) = e^{tA} X_0 \quad \text{for} \quad X_0 \in \mathbb{R}^n \]
Suppose \( X(t) \) is a solution of \( X' = AX \).

Define \( Z(t) := e^{-tA}X(t) \).

\[
Z'(t) = e^{-tA}(-A)X(t) + e^{-tA}X'(t) \\
= e^{-tA}(-A)X(t) + e^{-tA}AX(t) \\
= 0
\]

Therefore \( Z(t) = Z(0) = X(0) \) for all \( t \).

Hence \( e^{-tA}X(t) = X(0) \implies X(t) = e^{tA}X(0) \),

Lemma

for \( X(0) \in \mathbb{R}^n \).

Wait... it was that easy to find the general solution in the \( n \)-dimensional case...

The catch: what is \( e^{tA} \)?

is there any hope of reducing this infinite series to a finite sum?

To write \( e^{tA} \) more simply, we will use the structure of \( A \).

Jordan Canonical Form
Thm: For any $A \in \text{Mn}(\mathbb{R})$, there exists $T \in \text{Mn}(\mathbb{R})$ invertible so that

$$T^{-1}AT = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_e \end{pmatrix}$$

where each $B_i$ is a square matrix of one of the following forms

(i) $\begin{pmatrix} \lambda & 1 \\ & \ddots & 1 \\ & & \lambda \end{pmatrix}$  
(ii) $\begin{pmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ & & \ddots & \ddots \\ 10 & 01 & \ddots & \alpha \beta \\ 01 & \ddots & \ddots & -\beta \alpha \end{pmatrix}$

For $\lambda, \alpha, \beta \in \mathbb{R}$, $\beta \neq 0$.

• For each real eigenvalue $\lambda$ of $A$ with multiplicity $m$,
  - For each (linearly independent) eigenvector $V_i$ of $\lambda$,
    we get a $k_i \times k_i$ block of the form (i),
    where $k_i \in [1, m]$ is the largest integer
    such that $(A - \lambda I)^{k_i-1} \mathbf{w} = V_i$ for some general eigenvector $\mathbf{w}$. 

\[ \sum_{i} k_i = m \]  
\[ \text{if } V_i \text{ is an eigenvector of } \lambda \]

- For each pair of complex eigenvalues \( \alpha + i\beta \) of multiplicity \( m \),
- For each (linearly independent, complex) eigenvector \( V_i \) of \( \alpha + i\beta \), we get a \( 2k \times 2k \) block of the form (ii), where \( k_i \in [1, m] \) is the largest integer s.t. \( (A - \lambda \mathbb{I})^{k_i-1} W = V_i \) for some general eigenvector \( W \).

\[ \sum_{j} k_j = m \]  
\[ \text{if } V_j \text{ is an eigenvector of } \alpha + i\beta \]

**Proof:** Graduate level algebra (or 108?)

**Remark:**
- The ordering of the blocks is not unique.
- \( 1 \times 1 \) block of the form (i) is \( \begin{bmatrix} \alpha \end{bmatrix} \)
- \( 2 \times 2 \) block of the form (ii) is \( \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \)
- We consider each complex eigenvalue together with its conjugate \( \overline{\lambda} \).
Rmk: Note that the sizes of all the blocks add up to \( n \)

\[
\sum \sum k_i + \sum 2k_i = n
\]

Cor: Given \( A \in \text{Mn}(\mathbb{R}) \) and \( \lambda \) is an eigenvalue w/ multiplicity \( m \) and \( m \) linearly independent eigenvectors, then the JCF of \( A \) has \( m \)
(i) 1\times 1 \text{ blocks of the form } [\lambda], \text{ if } \lambda \in \mathbb{R}
(ii) 2\times 2 \text{ blocks of the form } \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix}, \text{ if } \lambda \in \mathbb{C}, \lambda \neq \bar{\lambda}

\( \bar{\lambda} \) By Thm, for each eigenvector \( \mathbf{v}_i \) of \( \lambda \),

\[
\sum k_i = 1 \quad \text{and} \quad \sum \sum k_i = m
\]

By hypothesis \( \forall i : \mathbf{v}_i \) is an eigenvector \( \exists | = m \)

Thus, \( k_i = 1 \) for all \( i \).