Recall: Linearization

For $F: \mathbb{R}^n \to \mathbb{R}^n$, $F \in C^2(E)$, $E \subseteq \mathbb{R}^n$ (open)
if $x, x_0 \in K \subseteq E$ compact

$|F(x) - (F(x_0) + DF(x_0)(x - x_0))| \leq \frac{1}{2} C |x - x_0|^2$

$C = \max_{y \in K} \left| \frac{\partial^2 F_i}{\partial x_j \partial x_k}(y) \right|

Thus, if we want to understand the behavior of

$X' = F(X)$

near an equilibrium point $X_*$, (most of the time)
it is enough to study

$(X - X_*)' = X' = F(X) \approx F(X_*) + DF(x_*)(X - X_*)$

Def: The linearized system at an equilibrium point $X_*$ is

$U' = DF(X_*)U$.

Last time:
- Example where linearization correctly predicted behavior of a nonlinear system near sink + saddle
Example where linearization incorrectly predicted a center where nonlinear system had a spiral sink/source.

Def: $X^* \in \mathbb{R}^n$ is an equilibrium point of $X'=F(x)$ if $F(X^*) = 0$. $X^*$ is a hyperbolic equilibrium point if, in addition, none of the evolues of $DF(X^*)$ have zero real part.

**Major Result #8**

Thm: (Linearization Theorem / Hartman-Grobman Theorem)
Given $E \subset \mathbb{R}^n$ (open) and $F \in C^1(E)$, suppose $X^*$ is a hyperbolic equilibrium point. Then the system $X'=F(X)$ is topologically equivalent to the linearized system in a neighborhood of $X^*$, aka "topologically conjugate."

What does it mean to be topologically equivalent?

Quick detour: flow maps

Def (flow map): A flow map $\phi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function satisfying
(i) $\phi(0, X_0) = X_0$ and $X_0 \in \mathbb{R}^n$ implies $\phi_0$ is the identity map
(ii) $\phi(t, \phi(s, X_0)) = \phi(t+s, X_0)$ implies $\phi_t \circ \phi_s = \phi_{t+s}$
We abbreviate $\Phi(t, x)$ by $\Phi_t(x)$ and think of $\Phi_t(\cdot)$ as a function from $\mathbb{R}^m$ to $\mathbb{R}^m$.

Rmk: Any IVP \( \{ x' = F(x), \ x(0) = x_0 \} \) has a corresponding flow map, where $\Phi(t, x_0) = X(t)$, i.e. the solution of IVP at time $t$.

Rmk: For any $t \in \mathbb{R}$, $\Phi_t(\cdot)$ is invertible and $\Phi_t^{-1}(\cdot) = \Phi_{-t}(\cdot)$.

Back to what it means for two dynamical systems to be topologically equivalent...

**Def (homeomorphism):** Given $E_1, E_2 \subseteq \mathbb{R}^m$ (open), a **homeomorphism** from $E_1$ to $E_2$ is a function $h : E_1 \to E_2$ such that

(i) $h$ is invertible, $h^{-1} : E_2 \to E_1$
(ii) $h$ is cts
(iii) $h^{-1}$ is cts

**Ex:** $E_1 = E_2 = (0, +\infty)$, $h(x) = x^2$, $h^{-1}(x) = \sqrt{x}$

This is a homeomorphism $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}$, $h(x) = x^2$

Not invertible, so not a homeomorphism.
**Def:** (topologically equivalent): Given $E_1, E_2 \subseteq \mathbb{R}^n$ (open), $F \in C^1(E_1)$, $G \in C^1(E_2)$, consider the systems $X' = F(X)$ and $Y' = G(Y)$ with corresponding flow maps $\phi_F$ and $\phi_G$.

These systems are topologically equivalent if there exists a homeomorphism $h : E_1 \to E_2$ s.t. for all $x_0 \in E_1$, there exists $a > 0$ with

$$\phi_G(t, h(x_0)) = h(\phi_F(t, x_0)) \text{ for all } t \in (-a, a).$$

**Ex:** $h((x, y)) = (-y, x)$.

**Fact:** For any $T \in M_n(\mathbb{R})$ that is invertible, $h(X) = T X$ is a homeomorphism. $h^{-1}(X) = T^{-1} X$.

Consequently, if two systems are equivalent up to coordinate change, they are topologically equivalent.
Rmk: If \( X' = F(X) \) and \( Y' = G(Y) \) are topologically equivalent via a homeomorphism \( h \), then

- \( X_\ast \) is a equilibrium point of \( X' = F(X) \)
  iff \( h(X_\ast) \) is an equilibrium point of \( Y' = G(Y) \)
- \( \chi(t) \rightarrow X_\ast \) iff \( Y(t) = h(\chi(t)) \rightarrow h(X_\ast) \)

Rmk: Intuitively, two systems are topologically equivalent if there exists a homeomorphism \( h \) so that trajectories of one system map onto trajectories of other system and the direction of time is preserved.

Consequently, the phase portrait of one system is a "distorted version" of the phase portrait of the other system, where bending and reflecting are allowed, but not ripping.

Rmk: In Hartman-Grobman, we only know \( X' = F(X) \) is topologically equivalent to linearized system in a neighborhood of \( X_\ast \), that is there exists \( E_1 \) (open) containing \( X_\ast \) and \( E_2 \) (open) containing the origin s.t. the systems restricted to these sets are top. equiv.
Stability of Equilibrium Points

Consider IVP \( \{ x' = f(x) \} \)

Let \( \phi_t(x_0) \) denote the flow

**Def (stable equilibrium point):** An equilibrium point \( x^* \) is (Lyapunov) stable if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) s.t.

\[
\forall x_0 \in B_\delta(x^*) \Rightarrow \phi_t(x_0) \in B_\varepsilon(x^*) \ \forall t \geq 0
\]

\[
|x_0 - x^*| < \delta \quad \Rightarrow \quad |x(t) - x^*| < \varepsilon, \ \forall t \geq 0
\]

Intuitively, \( x^* \) is stable if trajectories that start off sufficiently close to \( x^* \) remain close for all time.

**Def (asymptotically stable):** A stable equilibrium point is asymptotically stable if there exists \( \delta > 0 \) s.t.

\[
\forall x_0 \in B_\delta(x^*) \Rightarrow \lim_{t \to \infty} \phi_t(x_0) = x^*
\]

\[
|x_0 - x^*| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} |x(t)| = x^*
\]
Ex: $X' = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} X$, $a \in \mathbb{R}$

- $|a| < 1$ \quad A.S.
- $a = -1$ \quad A.S.
- $-1 < a < 0$ \quad A.S.

- $a = 0$ \quad S. (not A.S.)
- $a > 0$ \quad not S.