Math 119a, Fall 2018
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Crashers sign in

Course website:
math.ucsb.edu/~kcraig/math/math_119a
(No Gauchospace)

Course goals
4. Mathematical modelling
4. When to use explicit formulas vs. (rigorous) qualitative analysis

Dynamical systems: Systems that evolve in time according to a fixed rule
(population growth, swinging pendulum, flashing fireflies)

Differential equation: An equation that expresses a relationship between a function and its derivatives.
Ex: (logistic equation) \( x'(t) = ax(t)(1-\frac{x(t)}{N}) \)
\( a, N > 0 \)

Biological interpretation: \( \frac{x'}{x} = a(1-\frac{x}{N}) \)
- per capita growth rate
- rate of growth for \( x \) small
- \( x(t) = \) population of organism at time \( t \)

Properties of logistic equation:
- ordinary differential eqn, since \( x(t) \) is a fn of one variable
- first order, since eqn only depends on first derivative of \( x \)
- autonomous, since the RHS \( f(x)=ax(1-\frac{x}{N}) \) only depends on \( x \) (not \( t \))
- nonlinear, since \( f(x) \) is nonlinear
  (linear fn: \( f(x)=mx \), affine fn: \( f(x)=mx+b \))

Solving an ODE means finding a fn \( x(t) \) that satisfies the eqn.

Sometimes, you can find an explicit formula for all solutions of an ODE. This is the general solution.
Ex: Solve logistic eqn via sep of variables.

\[ \frac{dx}{dt} = ax(1-\frac{x}{N}) = \frac{a}{N} x(N-x) \]

\[ A(N-x) + Bx = 1, \quad A = \frac{1}{N}, \quad B = \frac{1}{N} \]

\[ \int \left( \frac{A}{x} + \frac{B}{N-x} \right) dx = \int \frac{dx}{x(N-x)} = \int \frac{a}{N} dt \]

\[ \frac{1}{N} \log \left( \frac{x}{N-x} \right) = \frac{1}{N} \log(x) - \frac{1}{N} \log(N-x) = \frac{a}{N} t + C \]

\[ \log \left( \frac{x}{N-x} \right) = at + NC \]

\[ \ln \left( \frac{x}{N-x} \right) = ke^{at} \quad (k=e^{NC} > 0) \]

\[ (1+ke^{at}) x = Nke^{at} \]

\[ x = \frac{Nke^{at}}{1+ke^{at}} \quad (\star) \]

You can check by direct computation that \( x' = ax \left( 1 - \frac{x}{N} \right) \).

Typically, want to find how soln's behave for specific initial data.
Ex: Solve \( \begin{cases} x' = ax(1 - \frac{x}{N}) & (IVP) \\ x(0) = x_0 \end{cases} \)

We know \( x(t) = \frac{N ke^{at}}{1 + ke^{at}} \) is a sol'n \( \forall k \in \mathbb{R} \).

Want: \( x(0) = Nk = x_0 \iff k(N - x_0) = x_0 \iff k = \frac{x_0}{N - x_0} \)

Thus, \( x(t) = \frac{N \left( \frac{x_0}{N - x_0} \right) e^{at}}{1 + \left( \frac{x_0}{N - x_0} \right) e^{at}} \) solves (IVP).

Another approach to understanding logistic eqn: qualitative analysis via phase portrait

\( x \) in one dimension

"phase line"
Ex: \( x' = ax(1 - \frac{x}{N}) \), \( a, N > 0 \)

\[ f(x) = ax(1 - \frac{x}{N}) + f(x) \]

Phase portrait:

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\[ 0 \quad \uparrow \quad 2 \]
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“source”

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\[ N \quad \uparrow \quad 2 \]
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“sink”

**Moral:** For any ODE of the form \( x' = f(x) \),

\[ f(x_0) > 0 \] \hspace{1cm} \text{increase} \text{ at } x_0

\[ f(x_0) = 0 \] \hspace{1cm} \text{solutions don't move} \text{ at } x_0

\[ f(x_0) < 0 \] \hspace{1cm} \text{decrease} \text{ at } x_0

Using phase portrait, we can sketch a few sol’ns:

\[ x(t) \]

\[ x(t) = N \text{ (sink, solid line)} \]

\[ x(t) = 0 \text{ (source, dashed line)} \]

For all \( x_0 > 0 \), population tends towards carrying capacity \( N \).
Def: For any ODE of the form $x' = f(x)$,

- $x(t) = x_0$ is an equilibrium soln if $f(x_0) = 0$. In this case, $x_0$ is an equilibrium point or fixed point.

- An equilibrium point is a 
  - sink if $f$ is decreasing at $x_0$, draw $\text{source}$ increasing.

Ex: (Logistic model w/ harvesting)

$$x' = x(1-x) - h, \quad h > 0, \quad (a = N = 1)$$

$$f(x) = x(1-x) - h$$