**Recall:**

**Def:** For any ODE of the form \( x' = f(x) \),
- \( x(t) = x_0 \) is an equilibrium sol'n if \( f(x_0) = 0 \).
- In this case, \( x_0 \) is an equilibrium point or fixed point.

- For a one dim' ODE
  - an equilibrium point \( x_0 \) is a **sink** if \( f \) is **decreasing** at \( x_0 \), draw \( \leftarrow \rightarrow \)
  - source

**Ex:** (Logistic model with harvesting)

\[ x' = x(1-x) - h, \quad h > 0 \cup (a = N = 1) \]

\[ f(x) = x(1-x) - h \]

\[ f(x) = -x^2 + x - h \]

\[ f(x_0) = 0 \]

\[ \Leftrightarrow \quad x_0 = \frac{1 \pm \sqrt{1 - 4h}}{-2} \]

\[ = \frac{1}{2} \pm \frac{1}{a} - h \]
Def: For an ODE of the form \( x' = f(x, h) \) if the number or stability of equilibria change at some \( h = h_0 \), we say that a bifurcation occurs at \( h_0 \).

Warning: this is not a comprehensive list of all the things we call bifurcation.

For this example, a bifurcation occurs at \( h_0 = \frac{1}{4} \).

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Systems of ODEs

\[
x_1' = f_1(x_1, x_2, \ldots, x_n) \\
\vdots \\
x_n' = f_n(x_1, x_2, \ldots, x_n)
\]

\[
\begin{align*}
x_1' &= F(x) \\
x(t) &= \left( x_1(t), \ldots, x_n(t) \right) \\
F(x) &= \left( f_1(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n) \right)
\end{align*}
\]
Def: Given a solution $X(t)$, its trajectory is the set $X(t) = \{ Y \in \mathbb{R}^n : X(t) = Y \text{ for some } t \in \mathbb{T} \}$.

“image of $X(t)$”

The trajectory is a curve in phase space.

Planar systems
(two dim'l systems)

$x' = f(x, y)$
y' = g(x, y)

Planar linear systems

$x' = ax + by$
y' = cx + dy$

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

spoiler: the key properties of planar systems can be studied by considering linearizations of the system.

We can find an explicit general solution for any n-dim'l ODE $X' = AX$ by using the eigenvalues and eigenvectors of $A$.

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Linear Algebra Review
Systems of Linear Equations

$\text{Mn}(\mathbb{R}) = \{ n \times n \text{ matrices with real elements} \}$

Def: $A \in \text{Mn}(\mathbb{R})$ is invertible if $\exists \ C \in \text{Mn}(\mathbb{R})$ for which $AC = CA = I$. The matrix $C$ is the inverse of $A$ and is denoted by $A^{-1}$.

Prop: For any $A \in \text{Mn}(\mathbb{R})$, $V \in \mathbb{R}^n$, the system of linear eqns

$$AX = V$$

has a unique solution $X \in \mathbb{R}^n$ iff $A$ is invertible. The solution is $X = A^{-1}V$.

We care about invertibility of matrices b/c we need to solve systems of linear eqns to find eigenvalues and eigenvectors.

Cor: For any $A \in \text{Mn}(\mathbb{R})$, there exists: $X \neq 0$ s.t. $AX = 0 V$ iff $A$ is not invertible. $X$ is in the nullspace of $A$, i.e. $X \in \text{N}(A)$.
Def: For any $A \in M_n(\mathbb{R})$, its determinant is defined inductively by

\[ \det(A) = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det(A_{1k}) \]

where for $[a_{ij}] \in M_n(\mathbb{R})$,

\[ \det([a_{ij}]) = a_{11}. \]

Ex(0): For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$

Prop: (Key properties of determinant)
1. $A \in M_n(\mathbb{R})$ is invertible iff $\det(A) \neq 0$.
2. $A, B \in M_n(\mathbb{R})$, $\det(AB) = \det(A)\det(B)$.

Eigenvectors and Eigenvectors

Def: $v \in \mathbb{R}^n$ is an eigenvector of $A \in M_n(\mathbb{R})$ if $v \neq 0$ and $(A - \lambda I)v = 0$ for some $\lambda \in \mathbb{R}$. We say $\lambda$ is the eigenvalue corresponding to $v$, $AV = \lambda V$.

Note that if $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, then $w = \alpha v$, $\alpha \neq 0$, is also an eigenvector for $\lambda$: $Aw = \lambda w$. 
\[ A\mathbf{w} = A(\alpha \mathbf{v}) = \alpha AV = \alpha \lambda \mathbf{v} = \lambda \alpha \mathbf{v} = \lambda \mathbf{w} \]

**Lemma:** \( \lambda \) is an eigenvalue of \( A \in \text{M}_n(\mathbb{R}) \) iff
\[
\det(A - \lambda I) = 0.
\]

This is called the **characteristic polynomial** of \( A \). The **roots** of this polynomial are the **eigenvalues** of \( A \).