Recall:
Linear systems of ODEs \( X' = AX, \quad X: \mathbb{R} \to \mathbb{R}^n \)
\( A \in M_n(\mathbb{R}) \)

Strategy for finding general solution:

Observation 1: if \( \lambda \) is an eigenvalue of \( A \) with eigenvector \( V \),
then \( X(t) = e^{\lambda t} V \) solves \( X' = AX \).

"Linearity Principle"

Observation 2: if \( X_1(t), \ldots, X_k(t) \) are solutions of \( X' = AX \),
then for any \( \alpha_1, \ldots, \alpha_k \in \mathbb{R} \),
\( Y(t) = \alpha_1 X_1(t) + \cdots + \alpha_k X_k(t) \)
is also a solution.

Goal: How can we take nice solutions constructed from eigenvalues and eigenvectors
and combine them to get general solution?

Prop: Given a subspace \( S \subseteq \mathbb{R}^n \) with dimension
\( \dim(S) = k \),
any collection of \( k \) linearly independent vectors \( V_1, \ldots, V_k \) in \( S \) forms a basis for \( S \).

Furthermore, if \( \{V_1, \ldots, V_k\} \) forms a basis for \( S \),
then for any \( Y \in S \), \( \exists! \) choice of \( \alpha_1, \ldots, \alpha_k \) s.t.
\( Y = \alpha_1 V_1 + \cdots + \alpha_k V_k \).
Thm: Suppose $\lambda_1, \ldots, \lambda_k$ are distinct real values of $A \in \text{Mn}(\mathbb{R})$ corresponding to eigenvectors $V_1, \ldots, V_k$. Then $V_1, \ldots, V_k$ are lin indep.

Cor: Suppose $A \in \text{Mn}(\mathbb{R})$ had $n$ distinct real values. Then the corresponding eigenvectors $V_1, \ldots, V_n$ form a basis for $\mathbb{R}^n$.

We can use this to find a gen'l soln of $X' = AX$ whenever $A$ has $n$ distinct, real values.

**MAJOR RESULT #1**

Thm: (distinct real values): Suppose $A \in \text{Mn}(\mathbb{R})$ had $n$ real distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ corresponding to eigenvectors $V_1, \ldots, V_n$. Then the gen'l soln to $X' = AX$ is any soln of $X = AX$ is of the form $X(t) = e^{\lambda_1 t}V_1 + \ldots + e^{\lambda_n t}V_n$, for some $\lambda_i \in \mathbb{R}$.

Rmk: Equivalently, the above theorem shows $\mathcal{E} = \{ e^{\lambda_1 t}V_1 + \ldots + e^{\lambda_n t}V_n : \lambda_1, \ldots, \lambda_n \in \mathbb{R} \}$ is the set of all possible solns of $X' = AX$.

Fact: The gen'l soln of $x' = ax, a \in \mathbb{R}, x : \mathbb{R} \rightarrow \mathbb{R}$, is $x(t) = ke^{at}$ for $k \in \mathbb{R}$. 
Suppose \( Y(t) \) is a soln of \( \dot{X} = AX \). For each \( t \in \mathbb{R} \), \( Y(t) \in \mathbb{R}^n \). Since \( V_1, \ldots, V_n \) form a basis for \( \mathbb{R}^n \), there exist \( \beta_1(t), \ldots, \beta_n(t) \in \mathbb{R} \) so that \( Y(t) = \beta_1(t) V_1 + \ldots + \beta_n(t) V_n \).

Since \( Y(t) \) is a soln of \( \dot{X} = AX \), we have
\[
Y'(t) = \beta_1'(t) V_1 + \ldots + \beta_n'(t) V_n
\]
\[
A Y(t) = \beta_1(t) \lambda_1 V_1 + \ldots + \beta_n(t) \lambda_n V_n
\]
Since \( V_1, \ldots, V_n \) are lin indep, we must have
\[
\beta_1'(t) = \beta_1(t) \lambda_1, \ldots, \beta_n'(t) = \beta_n(t) \lambda_n
\]
By Fact, \( \beta_i(t) = \alpha_i e^{\lambda_i t} \) for some \( \alpha_i \in \mathbb{R} \).
Thus,
\[
Y(t) = \alpha_1 e^{\lambda_1 t} V_1 + \ldots + \alpha_n e^{\lambda_n t} V_n
\]

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Phase portraits for planar, linear systems

Real distinct eigenvalues

Suppose \( A \in M_2(\mathbb{R}) \) has evnns \( \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \neq \lambda_2 \).
Ex: \( A = (\lambda_1, 0), \) evaluate eigenvectors \( \lambda_1, [0] \)
\( \lambda_2, [i] \)
By prev thm, gen'd soln: \( X(t) = \alpha e^{\lambda_1 t} [0] + \beta e^{\lambda_2 t} [i], \)
\( \alpha, \beta \in \mathbb{R} \)

\text{Case 1:} \( \lambda_1 < 0 < \lambda_2 \) ("saddle")
\( X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \)
the only equilibrium point is the origin. We call this type of equilibrium ad saddle point.

What happens as \( t \to \mp \infty \)?
\[
\lim_{t \to \infty} (\alpha e^{\lambda_1 t} [0] + \beta e^{\lambda_2 t} [i]) - \beta e^{\lambda_2 t} [i] = 0
\]
"as \( t \to +\infty \), the gen'l soln is indistinguishable from the corresponding stable line solution"
\[
\lim_{t \to -\infty} (\alpha e^{\lambda_1 t} [0] + \beta e^{\lambda_2 t} [i]) - \alpha e^{\lambda_1 t} [1] = 0
\]
"as \( t \to -\infty \), the gen'l soln is indistinguishable from the corresp. unstable line soln"
Case 2: $0 < \lambda_1 < \lambda_2$