Lecture 8

Recall: Planar, linear systems of ODEs
\[ \mathbf{x}' = A\mathbf{x}, \quad \mathbf{x} : \mathbb{R} \to \mathbb{R}^2, \quad A \in \mathbb{M}_2(\mathbb{R}) \]

Case 1: distinct real evales

Case 2: complex evales

Case 3: repeated real evales, i.e. \( A \in \mathbb{R} \) is the only evale \( A \in \mathbb{M}_2(\mathbb{R}) \)

L) Case (i): there are two linearly indep evectors corresp to \( \lambda \)

L) Case (ii): there is only one evector corresp to \( \lambda \)

L) Case (iii): there are no evectors corresp to \( \lambda \)

Recall:

Thm (fundamental theorem of algebra): For any polynomial with real coeff
\[ p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n, \quad a_i \in \mathbb{R}, \]
we may factor \( p(x) \) completely over \( \mathbb{C} \):
\[ p(x) = a_0(x - r_1)^{m_1}(x - r_2)^{m_2} \ldots (x - r_k)^{m_k} \]
where \( r_i \in \mathbb{C}, m_i \in \mathbb{N}, m_1 + m_2 + \ldots + m_k = n \)
\[ \uparrow \text{(algebraic multiplicity of root)} \]
roots of polynomial

**Def (multiplicity of value):** Given \( A \in \mathbb{M}_n(\mathbb{R}) \),
we may write
\[
\det(A - \lambda I) = (\lambda - \lambda_1)^{m_1} \cdot (\lambda - \lambda_2)^{m_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{m_k}
\]
\[ \uparrow \text{eigenvalue} \]
\[ \uparrow \text{(algebraic multiplicity of eigenvalue)} \]

Thus, \( \lambda \) is a repeated real eigenvalue of \( A \in \mathbb{M}_n(\mathbb{R}) \)
iff it's an evalue with multiplicity 2.

**Ex:** \( A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \), \( a \in \mathbb{R} \)

evaluated: \( \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & 1 \\ 0 & a - \lambda \end{pmatrix} = (a - \lambda)^2 \)

Thus, \( \lambda = a \) is an evalue w/ multiplicity 2.

**Case (i):** \( \lambda \) is an evalue of \( A \in \mathbb{M}_n(\mathbb{R}) \) w/ mult. 2 and there are two lin indep vectors corresp to \( \lambda \).
By HW 4, Q5: This implies $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

Now, we can quickly find the general soln:

Suppose $x' = Ax$. Then...

$$x' = (A\mathbf{I}) x = \lambda x \iff \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \iff \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix} = e^{\lambda t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Therefore, the gen' soln is:

**Thm:** If $\lambda$ is a repeated real eigenvalue of $A \in \mathbb{M}_n(\mathbb{R})$ with two lin indep eigenvectors, then the gen' soln

$$x(t) = c_1 e^{\lambda t} \mathbf{v}_1(t) + c_2 e^{\lambda t} \mathbf{v}_2(t), \quad c_1, c_2 \in \mathbb{R}.$$  

**Phase portrait:**

- $\lambda < 0$: origin is a stable star, all solutions are line solutions (every vector is an eigenvector).
- $\lambda > 0$: origin is unstable star.
Case (i): Suppose \( \lambda \) is an evlue of \( A \in M_2(\mathbb{R}) \) with mult. 2 and \( V \) is the only evector.

By HW3, Q7: if \( A \in M_2(\mathbb{R}) \) has a repeated evlue \( \lambda \), then for any \( V \in \mathbb{R}^2 \):
- either \( V \) is an evector of \( A \)
- or \( W = (A-\lambda I)V \) is an evector of \( A \)

In fact, this is a special case of a more general theorem.

**Def (generalized evector):** Given \( A \in M_n(\mathbb{R}) \) with an evlue \( \lambda \) of multiplicity \( m \leq n \), a nonzero vec \( V \) is a generalized evector of \( \lambda \) if
\[
(A-\lambda I)^k V = 0 \text{ for } k \in \{1, \ldots, m\}
\]

**Rmk:**
- Every evlue has at least one evector, \( (A-\lambda I)^1 V = 0 \).
- Every evector is a generalized evector.

**Thm:** Given \( A \in M_n(\mathbb{R}) \) with evlue \( \lambda \) of multiplicity \( m \leq n \), there exists an integer \( k \leq m \) so that the dimension of \( \mathrm{null}(A-\lambda I)^k \) equals \( m \).
Case (i): $\lambda$ has two lin indep. vectors $\Rightarrow$
\[ n=2, \, m=2, \, k=1 \]

Case (ii): $\lambda$ only has one vector $\Rightarrow$
\[ n=2, \, m=2, \, k=2 \]

Thus, if we have an eigenvalue of multiplicity $m$, $\exists \, k$ so we can find an $m$-clim$^2$ basis for $\forall (A-\lambda I)^k$
$\Rightarrow$ we can find $m$ lin indep generalized eigenvectors.

Goal: combine these to find general soln to $X'=AX$.

**Thm:** Given $A \in \mathbb{M}_2(\mathbb{R})$ with an eigenvalue $\lambda$ of multiplicity $2$ and a single vector $V$, the gen'1 soln of $X'=AX$ is

\[ X(t) = c_1 e^{\lambda t} V + c_2 (te^{\lambda t} V + e^{\lambda t} W) \]

for $(A-\lambda I)W = V$. 


Ex: \( A = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix} \), \( a, b \in \mathbb{R} \), \( a \neq 0 \), \( a > 0 \)

Eigenvalues: \( \det(A - \lambda I) = (a - \lambda)^2 - b = 0 \)

\[ \lambda = a \pm \sqrt{b} \]

Eigenvectors:

\[
\begin{pmatrix} a - \lambda & 1 \\ b & a - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \pm \sqrt{b} & 1 \\ b & \mp \sqrt{b} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

\[
\mathbf{v} = \begin{bmatrix} 1 \\ \pm \sqrt{b} \end{bmatrix}
\]

\( a > 0 \)

\( b > 0 \) (distinct real eigenvalues)

Assume \( \sqrt{b} < a \)

- Corresponding to \( \lambda = a + \sqrt{b} \)

\( b = 0 \) (repeated real eigenvalue)

- Degenerate node

\( b < 0 \) (complex eigenvalues)

- Spiral source
To write down the gen'l solution in the case $b=0$, need to find our generalized evector $W$ s.t. $(A-\lambda I)W = V$.

\[
\begin{pmatrix}
 a-\lambda & 1 \\
 0 & a-\lambda \\
 0 & 1 \\
 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
 x \\
 y \\
 x \\
 y \\
\end{pmatrix}
=
\begin{pmatrix}
 1 \\
 0 \\
 1 \\
 0 \\
\end{pmatrix}
\]

Thus $[0,1]$ is a generalized evector, so the general solution

$X(t) = c_1 e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 (te^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{at} \begin{pmatrix} 0 \\ 1 \end{pmatrix})$, $c_1, c_2 \in \mathbb{R}$