This is a closed-book and closed-note examination. Calculators are not allowed. Please show your work in the space provided. Scratch paper is not permitted. If you continue a problem on the back of a page, please write “continued on back”. Partial credit will be given for partial answers. There are 4 questions for a total of 100 points. You have 1 hour and 15 minutes.

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Question 1 (25 points)

Consider the following planar linear system:

\[ \begin{align*}
    x' &= y \\
    y' &= -2x - 3y
\end{align*} \]

(a) Find the general solution of the linear system.

(b) Sketch the phase portrait and classify all equilibrium points.
(c) Given a solution \((x(t), y(t))\) of the planar linear system, prove that

\[
\lim_{t \to -\infty} (x(t), y(t)) - c(-1, 1)e^{-t} = 0, \quad \text{for } c = -2x(0) - y(0).
\]
Inspired by Strogatz’s modeling of romantic relationships using planar linear systems, we will now study a model of your relationship with your roommate. Let

\[ y(t) = \text{the amount you like your roommate at time } t, \]
\[ r(t) = \text{the amount your roommate likes you at time } t. \]

Suppose that the more your roommate likes you, the more you dislike your roommate. (Another heart to heart conversation? You just want to get some sleep!) On the other hand, your roommate’s feelings tend to mirror yours: the more you like your roommate, the more your roommate likes you back. Both you and your roommate are cautious, and neither of you wants want your feelings toward the other to change too quickly.

(a) Under what conditions on \( a, b \in \mathbb{R} \) does the following planar linear system model your dysfunctional roommate relationship?

\[ y' = -by - ar \]
\[ r' = ay - br. \]

(b) Find the general solution of the linear system.
(c) Sketch the phase portrait. Classify all equilibrium points.

(d) Suppose that you and your roommate initially like each other a positive amount. (After all, you agreed to live together in the first place.) What is the ultimate outcome of your feelings for each other?

(e) Now suppose that you and your roommate throw caution to the wind ($b = 0$). Sketch the phase portrait. Classify all equilibrium points.

(f) If $b = 0$ and you and your roommate initially like each other a positive amount, what is the ultimate outcome of your feelings for each other?
Consider the planar linear system $X' = AX$, where $A \in M_2(\mathbb{R})$ and $X : \mathbb{R} \rightarrow \mathbb{R}^2$. Suppose that $A$ has a complex eigenvalue $\lambda_1$ corresponding to eigenvector $V_1$. (Assume $\lambda_1$ has nonzero imaginary part.)

(a) Prove that $\{V_1, \overline{V_1}\}$ are linearly independent over $\mathbb{C}$, i.e.

if $c_1V_1 + c_2\overline{V_1} = 0$ for $c_1, c_2 \in \mathbb{C}$, then $c_1 = c_2 = 0$.

(b) Use part (a) to show that $\{\text{Re}(V_1), \text{Im}(V_1)\}$ are linearly independent over $\mathbb{R}$, i.e.

if $c_1\text{Re}(V_1) + c_2\text{Im}(V_1) = 0$ for $c_1, c_2 \in \mathbb{R}$, then $c_1 = c_2 = 0$. 
(c) Prove that for any $c_1, c_2 \in \mathbb{R}$,

$$Y(t) = c_1 \text{Re}(e^{\lambda_1 t} V_1) + c_2 \text{Im}(e^{\lambda_1 t} V_1) \quad (1)$$

is a solution of $X' = AX$.

(d) Prove that for any choice of initial data $X_0 \in \mathbb{R}^2$, there exists a unique solution of the form (1) satisfying $Y(0) = X_0$. 

Question 4 (25 points)

Lightning Round!
You do not need to show your work or justify your answers.

(a) Match each of the phase portraits to the linear system $X' = AX$, for one of the below choices of $A$.

A. $\begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix}$,  
B. $\begin{bmatrix} 3 & -2 \\ 5 & -2 \end{bmatrix}$,  
C. $\begin{bmatrix} -3 & -2 \\ 5 & 2 \end{bmatrix}$.

(b) Match each of the linear systems to one of the following vector fields.

D. $X' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} X$,  
E. $\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$,  
F. $\begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix}$.